THE AMERICAN MATHEMATICAL MONTHLY.

DEVOTED TO THE
SOLUTION OF PROBLEMS IN PURE AND APPLIED MATHEMATICS,
PAPERS ON MATHEMATICAL SUBJECTS, BIOGRAPHIES
OF NOTED MATHEMATICIANS, ETC.

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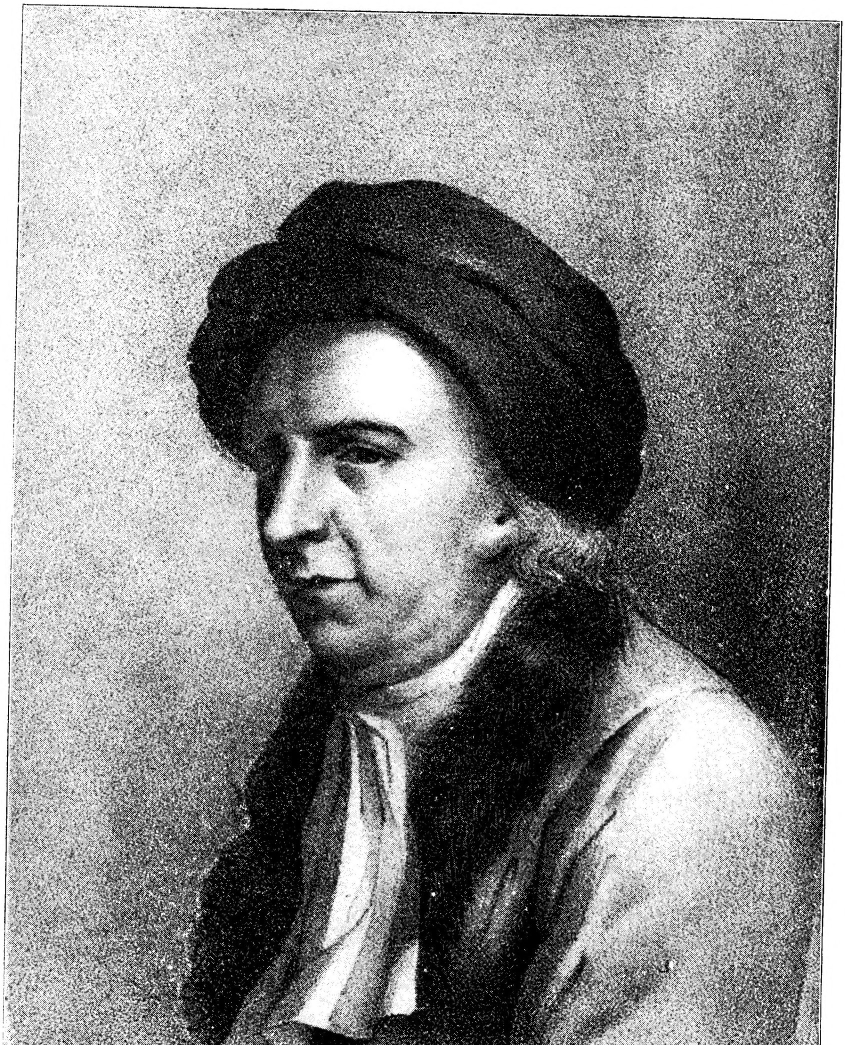

LEONHARD EULER

THE

AMERICAN

MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. JANUARY, 1898. No. 1.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from December Number.]

PROPOSITION XXXI. *Now I say there will be, of the aforesaid common perpendiculars in two distinct points, no determinate limit, such that under a smaller and smaller acute angle made at the point A, it would not always be possible to attain (in hypothesis of acute angle) to such a common perpendicular in two distinct points as is less than any assignable length R.*

PROOF. For in so far as the thing were otherwise; if from the point K (resume Fig. 30.) in BX assigned at any however great distance from the point B, a perpendicular KL is erected, to which from point A (by Euclid I. 12) the perpendicular AL is supposed let fall, KL ought to be greater than the length R.

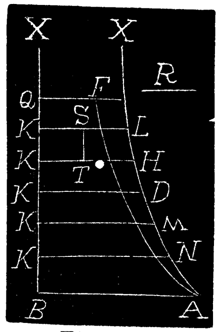

Fig. 30.

The reason is; because a higher point Q being assumed in this BX, from which is erected to BX the perpendicular QF, to which (by the same Euclid I. 12) a perpendicular AF is let fall, this again must anyhow not be less than the length R.

But KL (from Corollary to preceding Proposition) will be greater than QF. Therefore KL would be greater than the aforesaid length R. And so ever proceeding higher.

But now, if this however great KB is supposed divided (as in Proposition XXV) into portions KK, equal to the length R, and from these points K perpendiculars are erected, which meet AX in points H, D, M ; the angles at these points, toward the parts of the point L, will neither be right nor obtuse ; lest in some quadrilateral, as suppose $KMLK$, the four angles together should be equal to or greater than four rights, contrary to the hypothesis of acute angle, according to which we are proceeding. Therefore all such angles will be acute toward the parts of the point L ; and therefore in like manner all at these points obtuse toward the parts of the point A. Wherefore (from Corollary I to Proposition III) of the aforesaid perpendiculars the least will indeed be KL more remote from the base AB, the greatest KM nearer this base.

And of the remaining the nearer will be ever greater than the more remote.

Therefore (from the preceding Proposition XXV, and its Corollary) the four angles together of the quadrilateral $KHLK$ more remote from base AB will be greater than the four angles together of all the remaining quadrilaterals nearer to this base. Wherefore (as in Proposition XXV) the hypothesis of acute angle would be destroyed.

Therefore it holds, that of the aforesaid common perpendiculars in two distinct points there will be no determinate limit, such that under a smaller and smaller acute angle made at the point A, it would not always be possible to attain (in hypothesis of acute angle) to such a common perpendicular in two distinct points as may be less than any assigned length R.

Quod erat demonstrandum.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton, New Jersey.

IV.

PROOF OF LIE'S THEOREM THAT A ONE PARAMETER GROUP CONTAINS BUT ONE INFINITESIMAL TRANSFORMATION AND ITS CONVERSE THEOREM. EXAMPLES.

13. In the preceding paragraphs it has been shown by methods of proof due to LIE that every one parameter group with inverse transformations contains an infinitesimal transformation and conversely, every infinitesimal transformation generates a one parameter group. It is the purpose of this paragraph to present the proof of the theorem that the indefinite article "a" in these theorems can be replaced by the definite modifier "one and but one." The theorem

is necessary to a rigorous grounding of the fundamental details of the theory of the group of one parameter ; the proof is less simple than the proofs of the previous theorems because it makes use of an elementary theorem of the theory of functions.

Consider again the G_1

$$x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a). \tag{1}$$

We have found that every G_1 contains an infinitesimal transformation ; hence we may assume the existence of an infinitesimal transformation, say

$$x'=x+\xi(x, y)\delta t+\ldots\ldots, \quad y'=y+\eta(x, y)\delta t+\ldots\ldots, \tag{2}$$

belonging to the G_1, (1).

Let T_a be the transformation of the group (1) corresponding to the value a of the parameter and carrying the point (x, y) into the position (x_1, y_1). Let I be the infinitesimal transformation (2) of the group (1) and let it change the point (x_1, y_1) into the point (x_2, y_2) given by the equations

$$I \qquad x_2=x_1+\xi(x_1, y_1)\delta t+\ \ldots\ldots, \quad y_2=y_1+\eta(x_1, y_1)\delta t+\ \ldots\ . \tag{3}$$

The transformation equivalent to the product T_aI transforms the point (x, y) into the point (x_2, y_2) and is found by eliminating (x_1, y_1) from the equations (3) by means of the equations (1). This elimination partially effected gives

$$S\equiv T_aI\equiv T_{a+\delta a}\begin{cases} x_2=\varphi(x, y, a)+\xi(x_1, y_1)\delta t+\ldots\ldots, \\ y_2=\psi(x, y, a)+\eta(x_1, y_1)\delta t+\ldots\ldots, \end{cases} \tag{4}$$

where the x_1 and y_1 allowed to remain are to be expressed in terms of x, y, and a by means of the equations (1).

The equations (4) represent the transformation S which is equivalent to the successive application of T_a and I. Since T_a and I belong to the group (1), their product S belongs, by definition, to the group (1). Since I is an infinitesimal, the product, S, of T_a and I differs by an infinitesimal from T_a and accordingly the parameter of S has a value, $a+\delta a$, differing by an infinitesimal from the parameter, a, of T_a, where δa is an infinitesimal quantity of the same nature as δt in the equations (2). Further, we have proved that if a is the parameter of a transformation of a given group and a_1 the parameter of a second transformation of the same group, the parameter α of their product is a function of a and a_1 alone. The parameter of T_a is a, that of I is δt, and that of S, the product of T_a and I, is $a+\delta a$; hence $a+\delta a$ is a function of a and δt alone, *i. e.* δa depends on a and δt alone.

The transformation $T_{a+\delta a}$ which is equivalent to S and is a member of the group (1) has the form

$$x_2=\varphi(x, y, a+\delta a), \quad y_2=\psi(x, y, a+\delta a), \tag{5}$$

or developed in powers of δa,

$$
\begin{aligned}
x_2 &= \varphi(x, y, a) + \frac{\partial\varphi(x, y, a)}{\partial a}\delta a + \ldots\ldots, \\
y_2 &= \psi(x, y, a) + \frac{\partial\psi(x, y, a)}{\partial a}\delta a + \ldots\ldots.
\end{aligned}
\tag{6}
$$

Comparing these expressions (6) for x_2, y_2 with the forms given by the equations (4) we have

$$
\begin{cases}
\xi(x_1, y_1)\delta t + \ldots\ldots = \dfrac{\partial\varphi(x, y, a)}{\partial a}\delta a + \ldots\ldots, \\
\eta(x_1, y_1)\delta t + \ldots\ldots = \dfrac{\partial\psi(x, y, a)}{\partial a}\delta a + \ldots\ldots.
\end{cases}
\tag{7}
$$

In these equations (7), x_1 and y_1 are definite functions of x, y, and a, and conversely, x and y are definite functions of x_1, y_1, and a, given by the equations (1) ; δa and δt are infinitesimals and, as is shown above, δa is a function of a and δt ; the equations are true for all values of x, y, and a.

If in the equations (1) x and y are given any definite numerical values, x_1 and y_1 depend only on a. Hence if x and y are given any definite numerical values in the equations (7), these equations will express relations in δt, δa and a alone. These relations must of course agree with the one that expresses δa in terms of a and δt. Now we can choose the numbers x_1 and y_1 so that the first coefficients of one of the two relations, namely the quaatities

$$
\xi(x_1, y_1), \quad \frac{\partial\varphi(x, y, a)}{\partial a}, \tag{8}
$$

or the quantities

$$
\eta(x_1, y_1), \quad \frac{\partial\psi(x, y, a)}{\partial a}, \tag{9}
$$

do not vanish.*

Then the first or second equation (7) gives the relation between δa, δt and a in the form

---

*It is obvious that both of the quantities $\xi(x_1, y_1)$ $\eta(x_1, y_1)$ cannot vanish identically, else we should have no transformation. We may assume then that $\xi(x_1, y_1)$ does not vanish. It is clear then that $\dfrac{\partial\varphi(x, y, a)}{\partial a}$ cannot, in general, vanish ; for if it should vanish identically for all values of x and y, $\varphi(x, y, a)$ would be free from a and hence x would not be transformed at all, *i. e.* $\xi(x_1, y_1)$ would be identically zero ; but the latter is contrary to hypothesis.

$$u_1\delta t+u_2\delta t^2+\ldots\ldots=v_1\delta a+v_2\delta a^2+\ldots\ldots, \tag{10}$$

in which $u_1, u_2, \ldots\ldots, v_1, v_2, \ldots\ldots$ depend upon a. u_1, v_1 are both different from zero since they replace the quantities (8) or the quantities (9).

Now it is a theorem of the theory of functions that when two quantities δa and δt are related as in (10), δa may be developed in a power series of δt, whose first coefficient is not zero, *i. e.*

$$\delta a=w_1\delta t+w_2\delta t^2+\ldots\ldots,$$

where w_1 is not identically zero. The w_i are certain functions of a. Substituting this value of δa in the equations (7), conceiving x_1 and y_1 as variables again, we have

$$\begin{cases} \xi(x_1, y_1)\delta t+\ldots\ldots=\dfrac{\partial\varphi(x, y, a)}{\partial a}(w_1\delta t+w_2\delta t^2+\ldots\ldots)+\ldots\ldots, \\ \\ \eta(x_1, y_1)\delta t+\ldots\ldots=\dfrac{\partial\psi(x, y, a)}{\partial a}(w_1\delta t+w_2\delta t^2+\ldots\ldots)+\ldots\ldots. \end{cases} \tag{11}$$

Dividing through by δt and passing to the limit $\delta t=0$, these become

$$\begin{cases} \xi(x_1, y_1)=\dfrac{\partial\varphi(x, y, a)}{\partial a}w_1(a), \\ \\ \eta(x_1, y_1)=\dfrac{\partial\psi(x, y, a)}{\partial a}w_1(a), \end{cases} \tag{12}$$

The equations (1) solved for x and y give

$$x=\lambda(x_1, y_1, a), \quad y=\mu(x_1, y_1, a). \tag{13}$$

If these values of x and y be put in the equations (12) we have

$$\begin{cases} \xi(x_1, y_1)=X(x_1, y_1, a)w_1(a), \\ \eta(x, y_1)=Y(x_1, y_1, a)w_1(a). \end{cases} \tag{14}$$

The equations (14) must be true for all valoes of x_1, y_1 and a. Their left members do not contain a, hence their right members do not really contain a, but only in appearance, *i. e.* the functions X and Y have the form

$$X(x_1, y_1, a)\equiv\frac{A(x_1, y_1)}{w(a)}, \quad Y(x_1, y_1, a)\equiv\frac{B(x_1, y_1)}{w(a)}. \tag{15}$$

If we give now to the quantity a in the equations (14) a definite value* $\bar{a}$, the functions X and Y are changed into functions of x_1 and y_1 alone,

---

*For example, a solution of the equation $w(a)-1=0$ would be such a value of a that would reduce the functions X and Y to functions of x_1 and y_1 alone.

$$X(x_1, y_1, \bar{a}) \equiv \bar{X}(x_1, y_1), \quad Y(x_1, y_1, \bar{a}) \equiv \bar{Y}(x_1, y_1). \tag{16}$$

$w_1(a)$ becomes $w_1(\bar{a})$, which is a constant, since $\bar{a}$ is a definite number. Then, excepting this constant factor, the defining functions $\xi(x_1, y_1)$ and $\eta(x_1, y_1)$ of the infinitesimal transformation are completely determined, that is, if we call this constant k, we have

$$\xi(x_1, y_1) = k\,\bar{X}(x_1, y_1), \quad \eta(x_1, y_1) = k\,\bar{Y}(x_1, y_1). \tag{17}$$

This result obtains for *every* infinitesimal transformation (2) of the G_1 (1).

Any two infinitesimal transformations of the group (1), say

$$x' = x + \xi\delta t + \ldots\ldots, \quad y' = y + \eta\delta t + \ldots\ldots$$

and

$$x' = x + \bar{\xi}\delta t + \ldots\ldots, \quad y' = y + \bar{\eta}\delta t + \ldots\ldots \tag{18}$$

can differ in their terms of the first order only by a constant factor, since

$$\bar{\xi} \equiv k\xi, \quad \bar{\eta} \equiv k\eta.$$

Lie calls two such infinitesimal transformations, whose terms of the first order differ only by a constant factor, *dependent* infinitesimal transformations since they are essentially the same, for, inasmuch as δt was taken as an arbitrary infinitely small quantity, $k\delta t$ has the same meaning as δt.

In this manner Lie arrives at the theorem : *A one parameter group of the plane with inverse transformations contains one infinitesimal transformation and no more ; or, more accurately expressed, all infinitesimal transformations of a one parameter group agree in their terms of the first order excepting a constant factor.**

14. The factor $w_1(a)$ may be gotten rid of in the equations (12) by introducing a new parameter in the group (1) in place of a. If a_0 is the value of the parameter a which gives the identical transformation of the group (1), the new parameter, t, is given by

$$t = \int_{a_0}^{a} \frac{da}{w_1(a)}, \tag{19}$$

and the equations of the group become

$$x_1 = \Phi(x, y, t), \quad y_1 = \Psi(x, y, t). \tag{20}$$

Since $a = a_0$ makes $t = 0$ in (19), it is clear that in the form (20) the identical transformation, $x_1 = x$, $y_1 = y$, is given by the value of the parameter t=0. Hence the equations (12) may be written

---

*See Lie—*Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, bearbeitet und herausgegeben von Dr. Georg Scheffers, Leipzig, 1891, pp. 38 et seq.

$$
\left\{
\begin{aligned}
\xi(x_1, y_1) &= \frac{\partial \varphi(x, y, a)}{\partial a}\frac{da}{dt} \equiv \frac{\partial \Phi(x, y, t)}{\partial t}, \\
\eta(x_1, y_1) &= \frac{\partial f(x, y, a)}{\partial a}\frac{da}{dt} \equiv \frac{\partial \Psi(x, y, t)}{\partial t},
\end{aligned}
\right.
\tag{21}
$$

or
$$\xi(x_1, y_1) = \frac{dx_1}{dt}, \quad \eta(x_1, y_1) = \frac{dy_1}{dt}, \tag{22}$$

since x_1, y_1 are the functions (1) of a or the functions (20) of t.

Hence the equations (20), which are equivalent to the original equations (1) of the group, are the integral equations of the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1)} = \frac{dy_1}{\eta(x_1, y_1)} = dt, \tag{23}$$

with the initial conditions $x_1 = x$, $y_1 = y$, $t = 0$.

Since this simultaneous system and its integral equations are completely determined when the first members

$$x' = x + \xi(x, y)\delta t, \quad y' = y + \eta(x, y)\delta t,$$

of the infinitesimal transformation of the group are given, LIE has the theorem—*a one parameter group of the plane is completely defined by its infinitesimal transformation ; or otherwise expressed, every infinitesimal transformation belongs to one one parameter group of the plane and no more.*

This theorem and the preceding one are incorporated in the following second fundamental theorem of LIE's theory of the group of one parameter.

THEOREM : *Every one parameter group of the plane whose transformations are inverse in pairs contains one infinitesimal transformation and no more. Every infinitesimal transformation of the plane belongs to one one parameter group and to one only. The latter's transformations are inverse in pairs.*

The reader may find it to his advantage to work through the following examples carefully. They are in illustration of the theorems of the last two notes of this series and are drawn from LIE's lectures on differential equations.

1°. If the given infinitesimal transformation has the form

$$x_1 = x + x\delta t, \quad y_1 = y + y\delta t,$$

then
$$\xi(x, y) \equiv x, \quad \eta(x, y) \equiv y,$$

and the simultaneous system is

$$\frac{dx_1}{x_1} = \frac{dy_1}{y_1} = dt.$$

The integration of this simultaneous system gives

$$\log x_1 - \log x = \log y_1 - \log y = t,$$

or
$$x_1 = e^t x, \quad y_1 = e^t y.$$

These equations represent a one parameter group as may be readily verified directly. Any transformation of the group changes all abscissas and ordinates in the same ratio, *i. e.* the entire plane is proportionately increased or diminished from the origin of coördinates. $t=0$ gives the infinitesimal transformation of the group. By the exponential theorem we have

$$e^{\delta t} = 1 + \frac{\delta t}{1\,!} + \frac{\delta t^2}{2\,!} + \ldots\ldots;$$

hence the infinitesimal transformation of the group is

$$x_1 = (1 + \frac{\delta t}{1\,!} + \frac{\delta t^2}{2\,!} + \ldots\ldots)x, \quad y_1 = (1 + \frac{\delta t}{1\,!} + \frac{\delta t^2}{2\,!} + \ldots\ldots)y,$$

which agrees, in its terms of the first order, with the original infinitesimal transformation.

2°. The equations

$$x_1 = \sqrt{x^2 + xyt}, \quad y_1 = \frac{xy}{\sqrt{x^2 + xyt}}$$

represent a one parameter group, as appears in the following manner by making use again of the fundamental theorem of note III. We have clearly

$$x_1 y_1 = xy \quad \text{and} \quad \frac{x_1}{y_1} = \frac{x^2 + xyt}{xy} \equiv \frac{x}{y} + t.$$

These equations have the form of those in the theorem named if we put

$$\Omega(x, y) \equiv xy, \quad W(x, y) \equiv x/y.$$

$t=0$ corresponds to the identical transformation of the group, hence the infinitesimal transformation has the parameter $t=\delta t$. Since

$$\sqrt{x^2 + xyt} = x\sqrt{1 + \frac{y}{x}\delta t} = x(1 + \tfrac{1}{2}\frac{y}{x}\delta t + \ldots\ldots),$$

the infinitesimal transformation of the group is represented by tne equations

$$x_1 = x + \tfrac{1}{2}y\delta t + \ldots\ldots, \quad y_1 = y - \tfrac{1}{2}\frac{y^2}{x}\delta t + \ldots\ldots.$$

Hence,
$$\xi(x, y) \equiv \tfrac{1}{2}y, \quad \eta(x, y) \equiv -\tfrac{1}{2}\frac{y^2}{x},$$

and the finite equations of the group will appear again by integrating the simultaneous system

$$\frac{2dx_1}{y_1} = -\frac{2x_1 dy_1}{y_1^2} = dt.$$

3°. x_1 and y_1, the roots u of the quadratic equation

$$(u-x)(u-y)+t=0,$$

expressed in terms of x, y and t, define a G_1.

The equation may be written

$$u^2-(x+y)u+xy+t=0.$$

Then by an elementary theorem of the theory of algebraic equations,

$$x_1+y_1=x+y,$$

$$x_1 y_1=xy+t.$$

These are two equations of the form

$$\Omega(x_1, y_1)=\Omega(x, y), \quad W(x_1, y_1)-t=W(x, y),$$

and hence represent a group.

By actually solving the equations it may readily be shown that thatt he infinitesimal transformation of the group is given by the equations

$$x_1=x+\frac{\delta t}{y-x}+\ldots\ldots, \quad y_1=y+\frac{\delta t}{x-y}+\ldots\ldots.$$

4°. The equations

$$x_1=x+t, \quad y_1=\frac{xy-t}{x+t}$$

represent a G_1. In this case

$$\xi(x, y)\equiv, \quad \eta(x, y)\equiv-\frac{1+y}{x}.$$

The finite equations are integral equations of the simultaneous system

$$\frac{dx_1}{1} = -\frac{x_1 dy_1}{1+y_1} = dt.$$

The reader will have no trouble in verifying these results.

Princeton University, 6 January, 1898.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

84. Proposed by SYLVESTER ROBBINS, North Branch, New Jersey.

Show how to find sides, integral, fractional, and irrational, for twenty-four triangles, each one containing 330 square yards.

Partial solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics in Drury College, Springfield, Mo.

This is properly a problem of Diophantine Analysis, and should have been published under that department.

Let a, b, c, $a>b>c$, be three integers having no common divisor, and let them represent the sides of the triangle. Then the area of a triangle in terms of its sides is

$$\triangle=\tfrac{1}{4}\sqrt{2\Sigma a^2b^2-\Sigma a^4}.$$

Solving this equation for a, we have,

$$a=\sqrt{b^2+c^2\pm 2\sqrt{b^2c^2-4\triangle^2}}.$$

Let $\sqrt{b^2c^2-4\triangle^2}=bc-2\triangle v$. Then $bc=(\triangle/v)(v^2+1)$ where v can be any number so far as making the quantity under the radical a perfect square is concerned.

$\therefore\ a=\sqrt{b^2+c^2\pm2\triangle/v(v^2-1)}$.

If the area of a triangle is expressed by an integer the following principles can be easily proved:

(1) The sides can not all be even.

(2) The sides can not all be odd.

(3) No two sides can be even.

COROLLARY 1. Hence, of the three sides one must be even and the other two odd.

COROLLARY 2. The perimeter of the triangle is expressed by an even number.

If we assume b and c to represent odd integers, and since $bc=(\triangle/v)(v^2+1)$, we may find values of b and c by giving v such values as will make $(\triangle/v)(v^2+1)$ an odd integer. Then by taking such factors of this integer for values of b and c as will render $\sqrt{b^2+c^2+2(\triangle/v)(v^2+1)}$, a perfect square, we get integral values required. For example, if $v=2$, $b=33$, and $c=25$, gives $a=52$. But this does not furnish a ready means of writing out integral values.

If any one will furnish us a more expeditious method we shall be pleased to publish it in the next issue of the MONTHLY.

85. Proposed by E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

In turning a one-horse chaise within a ring of a certain diameter, it was observed that the outer wheel made two turns, while the inner wheel made but one. The wheels were each 4 feet high; and supposing them fixed at the distance of 5 feet on the axletree, what was the circumference of the track described by the outer wheel? From *Greenleaf's National Arithmetic*.

Solution by EDWIN R. ROBBINS, Master of Mathematics, Lawrenceville School, Lawrenceville, N. J.

Let r=radius, then $2\pi r$=circumference of inner track.

$r+5$=radius, then $2\pi(r+5)$=circumference of outer track.

From problem, $2(2\pi r)=2\pi(r+5)$, whence $r=5$.

Hence, outer track$=20\pi=62.83184+$ feet.

Also solved by *J. SCHEFFER, G. B. M. ZERR, F. R. HONEY, CHARLES C. CROSS, LEE WILCOX*, and *P. S. BERG.*

86. Proposed by EDGAR H. JOHNSON, Professor of Mathematics, Emory College, Oxford, Ga.

$\frac{1}{7}=.\dot{1}4285\dot{7}$; $\frac{1}{11}=.\dot{0}\dot{9}$; $\frac{1}{13}=.\dot{0}7692\dot{3}$; $\frac{1}{17}=.\dot{0}58823594117647$.

Observe that if the numbers forming the first half of the repetend be added respectively to the numbers forming the second half of the repetend, the sum is in every case 9. What is the general law of which these are special cases?

I. Solution by O. W. ANTHONY, M. Sc. Instructor in Mathematics in Boys' High School, New York City.

Put $\dfrac{1}{x}=\dfrac{R}{10^n}\left[1+\dfrac{1}{10^n}+\dfrac{1}{10^{2n}}+\dfrac{1}{10^{2n}}+\text{ etc.}\right]$, where R is the sequence of digits in the repetend and u the number of digits.

Call R_1 the first half of the repetend expressed in whole numbers, and R_2 the second half.

Then $R_1+R_2=999......=10^{\frac{1}{2}n}-1$.

Also $R=10^{\frac{1}{2}n}R_1+R_2,\ =10^{\frac{1}{2}n}R_1-R_1+10^{\frac{1}{2}n}-1,\ =(10^{\frac{1}{2}n}-1)(R_1+1)$.

$$\therefore\ \frac{1}{x}=\frac{(10^{\frac{1}{2}n}-1)(R_1+1)}{10^n}\cdot\frac{10^n}{10^n-1},\text{ or } x=\frac{10^{\frac{1}{2}n}+1}{R_1+1}.$$

This gives the law of formation when the first part of the repetend is R_1.

Then, if $R_1=142$, $x=\dfrac{10^3+1}{143}=7$.

To find all repetends of six figures obeying the law, we proceed as follows:

$x=\dfrac{10^3+1}{R_1+1}=\dfrac{1001}{R_1+1}=\dfrac{13\times11\times7}{R_1+1}$. We must select such values for R_1 as will give x integral values.

If $R_1=12$, $x=77$.
If $R_1=10$, $x=9$.
If $R_1=6$, $x=143$.
If $R_1=76$, $x=13$.
If $R_1=142$, $x=7$.

If $R_1=90$, $x=11$.
Then $\frac{1}{7}=.142857\ldots\ldots$
$\frac{1}{11}=.090909\ldots\ldots$
$\frac{1}{13}=.076923\ \ldots\ldots$
$\frac{1}{77}=.012987\ldots\ldots$
$\frac{1}{91}=.010989\ldots\ldots$
$\frac{1}{143}=.006993\ldots\ldots$

To find eight figured repetends obeying the law, we have,

$$x=\frac{10001}{R_1+1}.$$

To find ten figured repetends we have,

$$x=\frac{100001}{R_1+1},$$

and so on. In the last instance we may take as an illustration,

$R_1=10$, then $x=9091$, and $\frac{1}{9091}=.0001099989\ldots\ldots$

The problem reduces to resolving numbers into prime factors.

II. **Solution by J. SCHEFFER, A. M., Hagerstown, Md.**

There are some very peculiar properties connected with circulating decimals, the one pointed out in the propounded question being one of them. Of all the prime numbers 2 and 5 are the only ones that do not produce circulating decimals when divided into 1. When 1 is divided by a prime number N, 2 and 5 excepted, the resulting circulator contains either $N-1$ circulating decimals or a factor of $N-1$. In the former case the period is called full. Of all the prime numbers less than 100 7, 17, 19, 23, 29, 47, 59, 61, and 97 produce full periods, the periods containing respectively 6, 16, 18, 22, 28, 46, 58, 60, and 96 circulating decimals. All these have the property that the sum of any figure in the first half of the period and the corresponding figure in the second half is equal to 9. This remarkable property may be found thus :

Let N represent the prime number, a the first half of the period, and b the second, and let the period contain $2n$ figures. We have,

$$\frac{1}{N}=\frac{a}{10^m}+\frac{a}{10^{2m}}+\frac{b}{10^{4m}}+\ \ldots\ldots\ =\left(a.10^m+b\right)\left(\frac{1}{10^{2m}}+\frac{1}{10^{4m}}+\ \ldots\ldots\right)$$

$$=\frac{a.10^m+b}{10^{2m}-1}.\quad \text{But } \frac{1}{N}=\frac{a}{10^m}+\frac{N-1}{N.10^m},\ \text{or}$$

$$\frac{b}{10^{2m}}+\frac{a}{10^{3m}}+\frac{b}{10^{4m}}+\ \ldots\ldots\ =\frac{N-1}{N.10^m}.$$

The first part of this equation is

$$=\frac{b.10^m+a}{10^{3m}}+\frac{b.10^m+a}{10^{5m}}+\ \ldots\ldots\ =\left(b.10^m+a\right)\left(\frac{1}{10^{3m}}+\frac{1}{10^{5m}}+\ \ldots\ldots\right)$$

$$=\frac{b.10^m+a}{10^m}\cdot\frac{1}{10^{2m}-1};\quad \therefore \frac{b.10^m+a}{10^m}\cdot\frac{1}{10^{2m}-1}=\frac{N=1}{N.10^m},$$

$$\text{or } \frac{b.10^m + a}{10^{2m} - 1} = 1 - \frac{1}{N} = 1 - \frac{a.10^m + b}{10^{2m} - 1}.$$

By transposition $\dfrac{(a+b)10^m + (a+b)}{10^{2m} - 1} = 1$, or, $(a+b)(10^m + 1) = 10^{2m} - 1$, or. $a + b = 10^m - 1$, but the number $10^m - 1$ must necessarily contain m 9's only.

If the period is not a full period, but the period contains an even number of figures, the above law still holds good.

Another property of circulation is that when the number that produces the circulator is multiplied by the period, the resulting product consists of 9's only. For, designating the period by A, we have $\dfrac{1}{N} = A + \dfrac{1}{N.10^m}$.

$\therefore \dfrac{10^m - 1}{10^m} = AN$; but $10^m - 1$ contains only 9's.

A third property of circulation with full periods is, that, when $1/N$ is multiplied by any any integral number from 2 to $N-1$, the resulting periods will contain the same figures and in the same succession, only beginning with a different figure. This interesting property may be found as follows :

$$\text{Let } \frac{1}{N} = \frac{a}{10} + \frac{b}{10^2} + \frac{c}{10^3} + \frac{d}{10^4} + \ldots\ldots \frac{l}{10^{N-1}} + \frac{a}{10^N} + \frac{b}{10^{N+1}} + \ldots .$$

Let us, for instance, denote the sum of the remaining fractions after the third by $\dfrac{r}{N.10^3}$, we have $\dfrac{1}{N} = \dfrac{a}{10} + \dfrac{b}{10^2} + \dfrac{c}{10^3} + \dfrac{r}{N.10^3}$, whence,

$$\frac{r}{N.10^3} = \frac{1}{N} - \frac{a}{10} - \frac{b}{10^2} - \frac{c}{10^3} = \frac{d}{10^4} + \frac{e}{10^5} + \ldots \frac{l}{10^{N-1}} + \frac{a}{10^N} + \frac{b}{10^{N-1}} + \ldots$$

Multiplying by 10^3, we have,

$$\frac{r}{N} = \frac{d}{10} + \frac{e}{10^2} + \ldots\ldots \frac{l}{10^{N-4}} + \frac{a}{10^{N-3}} + \frac{b}{10^{N-2}} + \ldots\ldots,$$

which proves the proposition.

If M and N are two different prime numbers, the number of figures in the period of the circulator resulting from converting $1/MN$ into a decimal fraction is the least common multiple of the number of figures in the periods of the circulator resulting from converting $1/M$ and $1/N$ into decimal fractions. There are some more properties of circulation, but space forbids me to mention them.

NOTE. Prof. Nelson S. Roray calls attention to the error in the statement of No. 78; ''minus 126'' should be minus 125. The problem is found on page 344, Case II., example 4, Brooks' Higher Arithmetic. Dr. Brooks, in a letter to us, also called our attention to this error.

Profs. P. S. Berg and Chas. C. Cross sent in solutions to problem 83 too late for credit in the November number.

ALGEBRA.

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76. Proposed by E. B. ESCOTT; Fellow in Mathematics, University of Chicago, Chicage, Ill.

Prove the identities

$$2-\sqrt{2}=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}\cdots\cdots$$

$$\frac{5-\sqrt{5}}{2}=\tfrac{1}{1}+\tfrac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}\cdots\cdots$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let N be any number, then

$$N=\frac{Np^m}{p^m}\cdot\frac{r^m}{r^m}=\frac{p^m}{r^m}\left[1-\left(\frac{p^m-Nr^m}{p^m}\right)\right].$$

$$\therefore \sqrt[m]{N}=\frac{p}{r}\left[1-\frac{p^m-Nr^m}{p^m}\right]^{1/m}=\frac{p}{r}\left[1-\left(\frac{p^m-Nr^m}{mp^m}\right)\right.$$

$$-\frac{(m-1)}{1.2}\left(\frac{p^m-Nr^m}{mp^m}\right)^2-\frac{(m-1)(2m-1)}{1.2.3}\left(\frac{p^m-Nr^m}{mp^m}\right)^3$$

$$\left.-\frac{(m-1)(2m-1)(3m-1)}{1.2.3.4}\left(\frac{p^m-Nr^m}{mp^m}\right)^4-\text{ etc.}\right].$$

Let $m=2$ and $p^m-Nr^m=1$,

$$\therefore \sqrt{N}=\frac{p}{r}\left(1-\frac{1}{p^2}\right)^{\frac{1}{2}}=\frac{p}{r}\left[1-\tfrac{1}{2}\left(\frac{1}{p^2}\right)-\frac{1}{2^3}\left(\frac{1}{p^2}\right)^2-\frac{1}{2^4}\left(\frac{1}{p^2}\right)^3\right.$$

$$\left.-\frac{5}{2^7}\left(\frac{1}{p^2}\right)^4-\text{ etc.}\right]\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ (1).$$

Let $m=2$ and $p^m-Nr^m=4$,

$$\therefore \sqrt{N}=\frac{p}{r}\left(1-\frac{4}{p^2}\right)^{\frac{1}{2}}=\frac{p}{r}\left[1-\tfrac{1}{2}\left(\frac{4}{p^2}\right)-\frac{1}{1.2.2^2}\left(\frac{4}{p^2}\right)^2\right.$$

$$\left.-\frac{1.3}{1.2.3.2^3}\left(\frac{4}{p^2}\right)^3-\frac{1.3.5}{1.2.3.4.2^4}\left(\frac{4}{p^2}\right)^4-\text{ etc.}\right].$$

$$\therefore \sqrt{N}=\frac{p}{r}\left[1-\frac{2}{p^2}-\frac{2}{p^4}-\frac{4}{p^6}-\frac{2.5}{p^8}-\frac{4.7}{p^{10}}-\frac{3.4.7}{p^{12}}-\text{ etc.}\right]\ldots(2).$$

In (1) let $N=2$, $p=17$, $r=12$, $\therefore p^2-2r^2=1$.

$$\therefore \sqrt{2}=\tfrac{17}{12}\left(1-\frac{1}{2.17^2}-\frac{1}{2^3.17^4}-\frac{1}{2^4.17^6}-\frac{5}{2^7.17^8}-\text{ etc.}\right).$$

$$\therefore\ 2-\sqrt{2}=2-\tfrac{17}{12}+\frac{1}{2.12.17}+\frac{1}{2^3.12.17^3}+\frac{1}{2^4.12.17^5}+\frac{5}{2^7.12.17^7}+\text{ etc.}$$

$$2-\sqrt{2}=\tfrac{1}{2}+\tfrac{1}{12}+\frac{1}{2.12.17}+\frac{1}{2^3.12.17^3}+\frac{1}{2^4.12.17^5}+\frac{5}{2^7.12.17^7}+\text{ etc.}$$

$$=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\left(\frac{1}{2^4.3.17.577}-\frac{1}{2^4.3.17.577.578}\right)$$

$$+\left(\frac{1}{2^4.3.17.577.578}-\frac{1}{2^4.3.17.577.578^2}\right)$$

$$+\left(\frac{1}{2^4.3.17.577.578^2}+\frac{1}{2^5.3.17.577.665857}-\right)+\text{ etc.}$$

$$\therefore\ 2-\sqrt{2}=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}+\frac{1}{2^5.3.17.577.665857}+\ldots\ldots$$

In (2) let $N=5$, $p=7$, $r=3$, $\therefore\ p^2-5r^2=4$.

$$\therefore\ \sqrt{5}=\tfrac{7}{3}\left(1-\frac{2}{7^2}-\frac{2}{7^4}-\frac{4}{7^6}-\frac{2.5}{7^8}-\frac{4.7}{7^{10}}-\text{ etc.}\right)$$

$$\frac{5}{\sqrt{2}}=\tfrac{6}{7}-\frac{1}{3.7}-\frac{1}{3.7^3}-\frac{2}{3.7^5}-\frac{5}{3.7^7}-\frac{2}{3.7^8}-\text{ etc.}$$

$$\frac{5-\sqrt{5}}{2}=\tfrac{5}{2}-\tfrac{7}{6}+\frac{1}{3.7}+\frac{1}{3.7^3}+\frac{2}{3.7^5}+\frac{5}{3.7^7}+\frac{2}{3.7^8}+\text{ etc. }=\tfrac{1}{1}+\tfrac{1}{3}+\frac{1}{3.7}$$

$$+\left(\frac{1}{3.7.47}-\frac{2}{3.7.47.49}\right)+\left(\frac{2}{3.7.47.49}-\frac{4}{3.7.47.49^2}\right)$$

$$+\left(\frac{4}{3.7.47.49^2}+\frac{1}{3.7.47.2207}-\right)+\text{ etc.}$$

$$\therefore\ \frac{5-\sqrt{5}}{2}=\tfrac{1}{1}+\tfrac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}\ldots\ldots$$

The above formulæ were used by Dr. Artemas Martin for extracting the root of numbers several years ago.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Since $2-\sqrt{2}$ is rather more than $\frac{1}{2}$, we can, confining ourselves to 4 terms, put $2-\sqrt{2}=\frac{1}{2}+\frac{1}{2^2x}+\frac{1}{2^3xy}+\frac{1}{2^4xyz}$, or by transposition, $\sqrt{2}=\frac{3}{2}-\frac{1}{4x}-\frac{1}{8xy}-\frac{1}{16xyz}$. Squaring and omitting the term $\frac{1}{256x^2y^2z^2}$, we get, after multiplying by $64x$,

$$16x=48-\frac{4}{x}+\frac{24}{y}-\frac{1}{xy^2}+\frac{12}{yz}-\frac{4}{xy}-\frac{2}{xyz}-\frac{1}{xy^2z},$$

whence $x=3$, and substituting this value and multiplying by $3y$, we get

$$4y=68-\frac{1}{y}+\frac{34}{z}-\frac{1}{yz};$$

$\therefore$ $y=17$, and substituting it, and multiplying by $17z$, we get $z=578-1=577$. If another term had been desired we would have annexed to the above series the term $\frac{1}{2^5 xyzt}$ and proceeded in the same manner, though for 4 terms, the series is sufficiently convergent to render the value of $2-\sqrt{2}$ correct for at least 8 decimals.

Since $\frac{5-\sqrt{5}}{2}$ is $=1+$ fraction, we put, restricting the series to 4 terms besides 1,

$$\frac{5-\sqrt{5}}{2}=1+\frac{1}{x}+\frac{1}{xy}+\frac{1}{xyz}+\frac{1}{xyzt},$$

$$\text{whence } \sqrt{5}=3-\frac{2}{x}-\frac{2}{xy}-\frac{2}{xyz}-\frac{2}{xyzt}.$$

Squaring and omitting the term $4/x^2y^2z^2t^2$, we have, transposing, suppressing the common factor 4, and multiplying by x,

$$x=3-1/x+3/y-1/xy^2-1/xy^2z^2+3/yz+3/yzt$$
$$-2/xy-2/xyz-2/xyzt-2/xy^2z-2/xy^2z^2t, \text{ whence } x=3.$$

Substituting this value, and multiplying by $3y$, we have

$$y=7-1/y-1/yz^2+7/z+7/zt-2/yz-2/yz^2t; \text{ whence } y=7.$$

Substituting, and multiplying by $7z$, we get

$$z=47-1/z+47/t-2/zt; \text{ whence } z=47.$$

Substituting, and multiplying by $47t$, we get $t=2207$.

Such series converge very rapidly. In German works they are called "Theilbruchreihen," signifying *partial fraction series.* Every common fraction may be converted into such a series by the following process.

Let the fraction be $\frac{15}{19}$.

$\therefore$ $\frac{15}{19}=\frac{1}{2}+1/2.2+1/2.2.7+1/2.2.7.10+1/2.2.7.10.19$,

so that, taking successively 1, 2, 3, 4 of these, we get as in continuous fractions, approximate values, as $\frac{1}{2}$, $\frac{3}{4}$, $\frac{11}{14}$, $\frac{221}{280}$. As in continuous fractions, the numerator of the difference of any two consecutive approximate fractions is always $=1$. Supposing the indeterminate equation $15x-19y=1$. Changing $\frac{15}{19}$ by the above method into such a

```
15)19(2
 --
 11)19(2
  --
   3)19(7
    --
    2)19(10
     --
      1)19(19
```

series, and take the last approximate fraction, viz, $\frac{221}{280}$, $x=280$, $y=221$, will furnish two values; also, the preceding fraction $\frac{11}{14}$, $x=14$, $y=11$.

77. **Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, N. H.**

Solve the equation, $(6x^2+x-3)^2-48^2=(x+15)^2$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and A. H. BELL, Hillsboro, Ill.

$(6x^2+x-3)^2-(48)^2=(x+15)^2$.

$\therefore\ 3x^4+x^3-3x^2-3x-210=0$.

$\therefore\ x_1=2.924412+$,

$x_2=-3.041623-$,

$x_3=2.804395\sqrt{-1}-.158044$.

$x_4=-2.804395\sqrt{-1}-.158044$.

II. Solution by J. MARCUS BOORMAN, Consultative Mechanician, Counsellor at Law, Inventor, Etc., Hewlett, Long Island, New York.

The *real* roots of $x^4+\frac{1}{3}x^3-x^2-x=70$ (the forms to which the given equation reduces) are:

$$\left.\begin{aligned} x&=+2.924412\ 149966\ 623189\ 6108(58211 \\ x_1&=-3.041622\ 694570\ 750484\ 61819(75892 \end{aligned}\right\}$$ true to "(" marks,—*probably* 25 decimals true.

Found thus: x^4 +	$\frac{1}{3}x^3$	$-1x^2$	$-1x$	-70
At sight $x=\pm3$(near), try	-3.04	$+8.24-$	-22.01	69.9504
	$-2.71-\times3$	$+7.24-$	-23.01	$0.04-$
Multiply by $-3.\ -.04$	$+.13$	21.72	69.03	error$+$*
	$.1084+$	$29-$	$.9204$	
	$+8.24-$	-22.01	69.9504	

*The root $\therefore$ is $-2.0316+$ nearly.

Like treatment by $+2.9$ gives 2.925 near true. I get the rest in a more concise way (shorter than Horner's).

The *resulting* equations of above roots are: (the coëfficients appear on *face of computation*)

$0=x^3+(+\text{root}+\frac{1}{3})x^2+8.526990\ 472861\ 28178\text{etc.}\ x+23.936434{,}541435{,}17396+$.

$0=x^3+(-\text{root}+\frac{1}{3})x^2+7.237594\ 384604\ 24939\text{etc.}\ x-23.914031{,}334310{,}10968+$.

$0=(5.966034,\ 84\text{etc})x^2+(1.289496{,}088257{,}03238+)x+46.950465{,}875795{,}28364+$.

Hence $x^2+0.216122,\ 788722x=-7.869626,\ 49385$ etc.

$\therefore\ x_1=-0.108\ \text{etc.}\pm\sqrt{-7.890+}$.

Shorter and *more accurate* than Horner's method, *especially* for 5th degree and 6th degree equations.

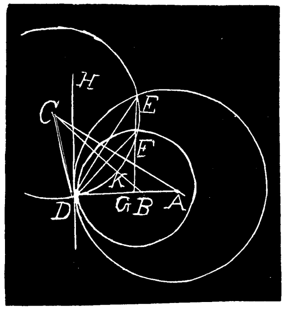
The other two roots are imaginary.

GEOMETRY.

80. Proposed by J. C. GREGG, Superintendent of Schools, Brazil, Ind.

One circle touches another internally, and a third circle whose radius is a mean proportional between their radii, passes through the point of contact. Prove that the other intersections of the third circle with the first two are in a line parallel to the common tangent of the first two. [From *Phillips and Fisher's Geometry.*]

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and the PROPOSER.



In the two triangles CDB and CDA, the angle CDG is common; also $DB : DC = DC : DA$.

$\therefore$ The two angles are similar and $\angle BCD = \angle CAD$, $\angle CBD = \angle ACD$. But $\angle DCB = \angle DEG$, both being measured by arc DK.

$\therefore$ $\angle DEG = \angle CAD$.

Since DE is perpendicular to CA, EG must be perpendicular to CA.

$\therefore$ HD and EG are parallel.

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa, and J. SCHEFFER, A. M., Hagerstown, Md.

Taking the common tangent of the first two circles as the axis of x and the common diameter as the axis of y and supposing the tangent of the third circle, through the origin to make the angle α with the axis of x, the equations of the circles are,

$$x^2+y^2-2ay=0,$$

$$x^2+y^2-2by=0, \text{ and}$$

$$x^2+y^2-2c(x\sin\alpha+y\cos\alpha)=0.$$

If the first and third circles intersect,

$$y=0 \text{ or } \frac{2ac^2\sin^2\alpha}{a^2+2ac\cos\alpha+c^2}.$$

If the second and third circles intersect,

$$y=0 \text{ or } \frac{2bc^2\sin^2\alpha}{b^2+2bc\cos\alpha+c^2}.$$

If the line through the second intersections is parallel to the axis of x the two values of y are equal.

$\therefore$ $ab^2+ac^2=a^2b+bc^2$. Whence $c^2=ab$.

If $c=\sqrt{ab}$, the distance y of the intersections from the common tangent is in both cases

$$\frac{2ab\sin^2\alpha}{a+\sqrt{ab}\cos\alpha+ab}. \qquad \text{Q. E. D.}$$

Also solved by *COOPER D. SCHMITT* and *CHARLES C. CROSS.*

81. **Proposed by CHAS. C. CROSS, Laytonsville, Md.**

A circle is drawn bisecting the lines joining the points of contact of the escribed circles with the sides produced. Another circle is drawn passing through the centers of the circles drawn tangent externally to the in-circle and internally to the sides of the triangle. Prove that the centers of these two circles, the incenter and the circumcenter are collinear.

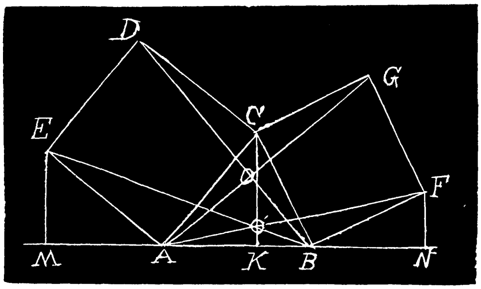
This problem is reprinted to correct an error in its enunciation. Inscribed is changed to *escribed*. EDITOR.

82. **Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, Cal.**

If the extremities of the base of a triangle be joined by straight lines to the exterior angles of squares constructed upon its two sides, the superior pair of lines thus drawn intersect at right angles; the inferior pair intersect at a point in a line drawn from the vertical angle perpendicular to the base.

I. **Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.; COOPER D. SCHMITT, A. M., Professor of Mathematics in the University of Tennessee, Knoxville, Tenn.; and CHAS. C. CROSS, Laytonsville, Md.**

1. In the two triangles BCD and ACG, we have $CG=CB$, $CD=CA$, $\angle BCD=\angle ACG$.



$\therefore \triangle BCD=\triangle ACG$ and $\angle BDC=\angle CAG$.

Hence, in quadrilateral $EAOD$, $\angle EAO+\angle EDO=$two right angles.

$\therefore \angle AED+\angle AOD=$two right angles, but $\angle AED=$a right angle and therefore AOD is a right angle.

$\therefore AG$ and BD are perpendicular to each other.

2. Let $AK=d$, $CK=f$. Then $BN=CK=f$, $FN=KB=b-d$, $AM=CK=f$, $EM=AK=d$.

$\therefore$ Equation to AF is $y=\dfrac{b-d}{b+f}x$, equation to BE is $y=-\dfrac{d}{b+f}(x-b)$.

These lines intersect at a distance $x=d$.

$\therefore AF$ and BE intersect on CK, the perpendicular from C on AB.

II. **Solution by HENRY HEATON, M. Sc., Atlantic, Iowa; J. SCHEFFER, A. M., Hagerstown, Md.; and J. C. GREGG, Superintendent of Schools, Brazil, Ind.**

Let $ACDE$ represent the square erected on the side AC, and $BCGF$ that on BC. Let O be the point of intersection of BD and AG, and O' that of BE and AF. In the triangles ACG and BCD, we have $CG=BC$, $AC=CD$, $\angle ACG=\angle BCD$. $\therefore \triangle ACG=\triangle BCD$. $\therefore \angle CDB=\angle CAG$, consequently the points A, O, C, D, are concyclic, and since $\angle DCA$ is a right angle, DOA must be a right angle.

Let EM and FN be the perpendicular let fall from E and F, respectively, upon AB produced. $O'H$ the perpendicular let fall from O' upon AB, and H' the foot of the perpendicular from C upon AB. We have

$$EM : O'H = MB : BH, \text{ and}$$
$$O'H : FN = AH : AN.$$
$$\therefore EM : FN = MB \times AH : AN \times BH,$$

but $MB = AM + AB = CH' + AB$, and $AN = BN + AB = CH' + AB$.

$\therefore MB = NA$, $\therefore EM : FN = AH : BH$, but $EM = AH'$, $NF = BH'$.

$\therefore AH' : BH' = AH : BH$.

$\therefore AH = AH'$, $BH = BH'$. Q. E. D.

CALCULUS.

63. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

What is the volume removed by boring an auger hole radius r through a right cylinder radius R, the center of the auger hole to pass at a distance c from the axis of the cylinder and inclined to the axis at an angle α?

I. Solution by the PROPOSER.

Let the axis of the cylinder whose radius is R coincide with the y-axis and let the axis of the cylinder whose radius is r intersect the z-axis at a distance c from the xy-plane, being parallel to the xy-plane and making an angle α with the y-axis. Pass a plane parallel to the xy-plane through the solid common to the two cylinders and at a distance z from the origin of coördinates. The intersection of this plane with the surface of the two cylinders forms a parallelogram, whose length is $2\sqrt{R^2-z^2}$ and whose width is $2\csc\alpha\sqrt{r^2-(z-c)^2}$. Hence its area is

$$\csc\alpha\sqrt{[r^2-(z-c)^2][R^2-z^2]}.$$

$\therefore V = \int_{c-r}^{c+r} \csc\alpha\sqrt{(R^2-z^2)[r^2-(z-c)^2]}dz$. Let $y = \dfrac{p+qy}{1+y}$, p and q to be determined from the conditions that the odd powers of z in the expansion under the radical shall vanish. From this condition we find $pq = R^2$ and

$$p+q = \frac{R^2-r^2+c^2}{c} \text{ or } \frac{R^2+r^2-c^2}{r}$$

according as $c > r$ or $< r$. From these two equations we find $p =$

$$\frac{R^2-r^2+c^2+\sqrt{(R^2-r^2+c^2)^2-4R^2c^2}}{2c} \text{ or } \frac{R^2+r^2-c^2+\sqrt{(R^2+r^2-c^2)^2-4R^2r^2}}{2r}$$

and q has the conjugate value of p. Making the substitution of $(p+qy)/(1+y)$, the values of p and q being replaced by their values in terms of R, r, and c as found above. The expression for the volume reduces to the following form :

$$V=-\csc\alpha\frac{\sqrt{(R^2-r^2+c^2)-4R^2c^2}}{c}[(R^2+c^2-r^2)^2$$

$$+(R^2+c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}-4R^2c^2]$$

$$[(R^2-c^2)^2-2c^2(R^2+r^2)+c^4-(R^2-c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}$$

$$\times\int\left[\left(1+\frac{(R^2+c^2-r^2)^2-(R^2+c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}-4R^2c^2}{(R^2+c^2-r^2)^2+(R^2+c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}-4R^2c^2}y^2\right)\right.$$

$$\left.\left(1+\frac{(R^2-c^2)^2-2c^2(R^2+r^2)+c^4-(R^2-c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}}{(R^2-c^2)^2-2c^2(R^2+r^2)+c^4+(R^2-c^2-r^2)\sqrt{(R^2+c^2-r^2)^2-4R^2c^2}}y^2\right)\right]^{\frac{1}{2}}$$

$$\times\frac{dy}{(1+y)^4},$$

which is of the form $k\int\sqrt{(1+my^2)(1+ny^2)}\,\frac{dy}{(1+y)^4}$. • Let $y=x/\sqrt{-m}$.

Then $V=-mk\sqrt{-m}\int\sqrt{[1-x^2][1-(n/m)x^2]}\,\frac{dx}{(x+\sqrt{-m})^4}$

$$=-m\sqrt{-m}\;k\int\frac{[1-x^2][1-(n/m)x^2]dx}{[x+\sqrt{-m}]^4\sqrt{[1-x^2][1-(n/m)x^2]}}$$

$$=-m\sqrt{-m}\;k\int\frac{\{1-[(m+n)/n]x^2+(n/m)x^4\}/[x+\sqrt{-m}]^4}{\sqrt{[1-x^2][1-(n/m)x^2]}}dx.$$

By multiplying both numerator and denominator of the numerator by

$$(x^4-mx^2+m^2)-\sqrt{-m}\,x(x^2-m),$$

we reduce the numerator to a function of x^2 only. From this point by some labor we can obtain the integral in terms of logarithmic and circular, and elliptic functions of the first, second, and third kinds. We have made this integration but the result is too long and complicated to put in print.

When $c<r$, c and r change places in the formula derived by the above solution.

COROLLARY 1. If $c=0$, and $\alpha=\frac{1}{2}\pi$, $V=\int_0^r\sqrt{(r^2-z^2)(R^2-x^2)}dz$.

Let $z=rx$, and $r/R=e$.

Then $V=2Rr^2\int_0^1\sqrt{(1-x^2)(1-e^2x^2)}dx$.

By following out the method indicated above we get

$$V=\tfrac{8}{3}R^3[(1+e^2)E(e)-(1-e^2)F(e)].$$

COROLLARY 2. By Prof. Henry Heaton, Atlantic, Iowa.

If $R=r$, $V=1/\sin\alpha\int_0^{R-\frac{1}{2}c}\sqrt{R^2-(x-\frac{1}{2}c)^2}\sqrt{R^2-(x+\frac{1}{2}c)^2}dx$, where $x+\frac{1}{2}c$ is the distance from axis of cylinder to the plane.

Put $x=(R-\frac{1}{2}c)\sin\theta$, and $(R-\frac{1}{2}c)/(R+\frac{1}{2}c)=e$.

Then $V=(2/\sin\alpha)(R-\frac{1}{2}c)^2(R+\frac{1}{2}c)\int_0^{\frac{1}{2}\pi}\cos^2\theta\sqrt{1-e^2\sin^2\theta}\,d\theta$

$$=(2/\sin\alpha)(R+\tfrac{1}{2}c)^3[(1+e^2)E(e,\tfrac{1}{2}\pi)-(1-e^2)F(e,\tfrac{1}{2}\pi)].$$

COROLLARY 3. By Prof. G. B. M. Zerr, Lebanon, Va., and Prof. C. W. M. Black, Wilbraham, Mass.

If $R=r$ and $c=0$, $V=(16r^3)/(3\sin\alpha)$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let $x^2+z^2=R^2$, be the equation to the cylinder.

Then $(x\cos\alpha+y\sin\alpha)^2+(z-c)^2=R^2$, is the equation to the auger hole.

The limits of y are,

$$y=\frac{\sqrt{R^2-(z-c)^2}-x\cos\alpha}{\sin\alpha}\text{ and }y=-\frac{\sqrt{R^2-(z-c)^2}+x\cos\alpha}{\sin\alpha}.$$

$$\therefore\ V=(4/\sin\alpha)\int_0^R\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\sqrt{R^2-(z-c)^2}\,dx\,dz$$

$$=(2/\sin\alpha)\int_0^R\Bigg[\left(\sqrt{R^2-x^2}-c\right)\sqrt{R^2-(\sqrt{R^2-x^2}-c)^2}$$

$$+R^2\sin^{-1}\left(\frac{\sqrt{R^2-x^2}-c}{R}\right)+\left(\sqrt{R^2-x^2}+c\right)\sqrt{R^2-(\sqrt{R^2-x^2}+c)^2}$$

$$+R^2\sin^{-1}\left(\frac{\sqrt{R^2-x^2}+c}{R}\right)\Bigg]dx.$$

This does not appear to be easy to reduce.

64. Proposed by E. S. LOOMIS, A. M., Ph. D., Professor of Mathematics in Cleveland West High School, Berea, Ohio.

Find the volume and surface generated by revolving about Y, the catenary

$$y=\tfrac{1}{2}a(e^{x/a}+e^{-x/a}),\text{ from }x=0\text{ to }x=a.$$

[*Osborne's Calculus*, page 255, example 8].

MECHANICS.

54. Proposed by C. H. WILSON, Poughkeepsie, N. Y.

A body slides from rest down a series of smooth inclined planes, whose total heights are h feet. Show that the velocity at the bottom is $\sqrt{(2gh)}$ feet per second. [From *Wright's Mechanics.*]

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let v_1, v_2, v_3, ... v_n be the velocities at the bottoms of the planes, h_1, h_2, h_3, ... h_n their respective heights.

$$\therefore\ v_1=\sqrt{2gh_1},\quad v_2=\sqrt{2gh_1+2gh_2}=\sqrt{v_1^2+2gh_2},$$

$$v_3=\sqrt{v_2^2+2gh_3}=\sqrt{2gh_1+2gh_2+2gh_3},$$

$$v_n=\sqrt{2gh_1+2gh_2+\ \ldots\ldots +2gh_n}=\sqrt{2g(h_1+h_2+h_3+\ldots\ldots+h_n}=\sqrt{2gh},$$

since $h_1+h_2+h_3+\ldots\ldots+h_n=h$.

Also solved by *HENRY HEATON*, *C. W. M. BLACK*, and *CHAS. C. CROSS*.

55. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Miss.

Three equal heavy spheres, each of weight W, are placed on a rough ground just not touching each other. A fourth sphere of weight nW is placed on the top touching all three. Show that there is equilibrium if the coefficient of friction between two spheres is greater than $\tan\frac{1}{2}\alpha$, and that between a sphere and the ground is greater than $\tan\frac{1}{2}\alpha n/(n+3)$, where α is the inclination to the vertical of the straight line joining the centers of the upper and one lower sphere.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, O.

Let α be the angle which the line of centers of the upper sphere and each of the lower makes with the vertical, R the reaction of the upper and each of the lower spheres, m the corresponding coefficient of friction, m_1 the coefficient for each lower sphere and the plane.

The system is kept at rest by the weights nW, W, acting vertically, R the reaction in direction of centers, mR, friction acting in the tangent through the point of contact of the upper and lower spheres, and $m_1(nW+3W)$ horizontally and inward.

For the equilibrium of the upper sphere, resolving vertically,

$$3R\cos\alpha+3Rm\sin\alpha=W\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

and for the lower, resolving horizontally,

$$\tfrac{1}{3}(nW+3W)m_1=R\sin\alpha-Rm\cos\alpha\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Also, $R=\frac{1}{3}nW\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$.

(1), (2), and (3) gives $m=\tan\frac{1}{2}\alpha$, $m_1=[n/(n+3)]\tan\frac{1}{2}\alpha$.

Also solved by *G. B. M. ZERR* and the *PROPOSER*. Their solutions will appear in next number.

DIOPHANTINE ANALYSIS.

55. **Proposed by O. W. ANTHONY, M. Sc. Instructor in Mathematics in Boys' High School, New York City.**

Construct a general Magic Square whose sum is $3m$.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

$$\begin{array}{lll} m+bn & m-(a+b)n & m+an \\ m+(a-b)n & m & m-(a-b)n \\ m-an & m+(a+b)n & m-bn \end{array}$$

II. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

I assume that the method of constructing Magic Squares is understood, and that they are made up of numbers in arithmetical progression.

Let p be the first term, q the common difference, and n the number of rows.

Then n^2=number of cells and of the numbers used; the sum of the series divided by n gives the sum of the numbers in each row=$3m$.

Or $\dfrac{n^2[r(n^2-1)+2p]}{2n}=3m$, or, $n[q(n^2-1)+2p)=6m$. . . . .(1).

As there are three unknown quantities, two of them may be assumed, and there will be a result for each assumption of both numbers.

Then $n=3$ and reducing we have $4q+p=m$; or $p=m-4q$, in which m may be any multiple of q greater than $4q$.

Take $m=5q$, then $q=m/5$, and $p=q$. In this m may be any number divisible by 5, and for every value we have a magic square, whose sum is $3m$.

Again, take $m=6q$, then $p=2q$, and $q=m/6$; m may be any number divisible by 6 ; and for every value we have a magic square, whose sum is $3m$.

In the same manner, we find that m may be any number divisible by any one of the numbers in the natural series from 5 upwards.

Now take $n=5$; substitute in (1), and we have by reducing $120q+10p=6m$. To simplify assume $m=5t$, and we have $12q+p=3t$; or $p=3t-12q$, in which t may be any number greater than $4q$.

Take $t=5q$; then $m=25q$, and $p=3q$, and $q=m/25$. So m may be any number divisible by 25.

In same manner, take $t=6q$, $7q$, etc., and we have $q=m/30$, $m/35$, etc., so that we may have m=any number divisible by 5 times any one of the natural numbers from 5 upwards, and as many magic squares of 5 rows.

In the same manner, we may obtain similar results by taking n=any other odd number.

Or, we can obtain a general solution directly from (1). Take $m=nt$, $n=2s+1$, substitute and reduce and we find $p=3t-2q(s^2+s)$, in which t may be $=q(s^2+s)$, or any multiple of it. Take $t=q(s^2+s)$; and then $p=q(s^2+s)$, (the first term) ; and $m=nt=q(2s+1)(s^2+s)$, and $q=m/[s(s+1)(2s+1)]$ (the common difference) ; in which s may be any integral, and $m=s(s+1)(2s+1)$, or any multiple of it.

But this formula does not enable us to obtain least values of p, q, and m, as n varies.

56. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.**

If $\phi(R)$ is the number of integers which are less than R and prime to it, and if y is prime to R, show that $y^{\Phi(R)}-1\equiv0(\text{mod}.R)$.

Solution by the PROPOSER, and J. O. MAHONEY, B. E., M. Sc., Lynnville, Tenn.

Let $1, m, n, p, \ldots\ldots(R-1)$ denote the $\phi(R)$ numbers less than R and prime to it ; now y can be any one of those numbers.

$\therefore y, my, ny, py, \ldots\ldots(R-1)y$ are all prime to R and all different.

There are $\phi(R)$ of such products and since when these products are divided by R the remainders are all prime to R and all different, the $\phi(R)$ remainders must be $1, m, n, p, \ldots\ldots(R-1)$ though not necessarily in this order.

$\therefore y.my.ny.py\ldots\ldots(R-1)y$ must differ from $1.m.n.p\ldots\ldots(R-1)$ by a multiple of R.

$\therefore \{y^{\Phi(R)}-1\}mnp\ldots\ldots(R-1)=$a multiple of R.

But $mnp\ldots\ldots(R-1)$ is prime to R.

$\therefore y^{\Phi(R)}-1\equiv0(\text{mod}.R)$.

57. **Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Maine.**

Each of *five* of the digits may be the terminal figure of a perfect integral square. Each of *eighteen* combinations of two digits may be the *two* terminal figures of an integral square. Each of *one hundred and nineteen* combinations of three digits may be the *three* terminal figures of an integral square. *Under these conditions*, what is the greatest number of arrangements of the nine digits, all taken together, whose three terminal figures shall be those of a square number ?

No solution of this difficult problem has been received. Can any of our readers furnish the desired solution ? EDITOR.

MISCELLANEOUS.

53. **Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pa.**

(a) What is the highest north latitude in which the Sun will shine in at the north window of a building at least once in a year ?

(b) How many days will it shine in at the north window of a building in latitude 41° N. ?

Note by SAMUEL HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

Whenever the Sun, or any part of it, is north of the prime vertical, it must then shine on the north side of buildings. From the time of vernal equinox, to the autumnal equinox, the Sun will be north of the prime vertical during some part of every day, and will shine on the north side of buildings some part of *every* day for about half a year, and in *all* latitudes north of the equator. Hence the answer for (a) is 90° N. latitude, and for (b) 186 days, but if the Sun's upper

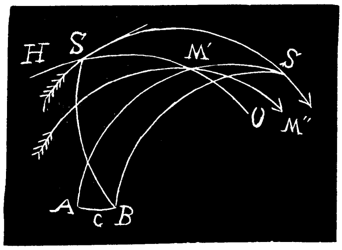
limb, and refraction, be considered the days will be 187 or 188. The answer for 41° N. is *good* for *any* other latitude north, while the problem seems to imply that an answer for 41° is different for other latitudes.

54. Proposed by **S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.**

On latitude 40° N.$=\lambda$, when the Moon's declination is $5^\circ\ 23'$ N.$=\delta$, and the Sun's $9^\circ\ 52'$ S.$=-\delta$, how long after sunset will the two horns or cusps of the Moon's crescent (recently new) set at the same moment, the crescent with its back *down* having touched the horizon first? Semi-diameters, refraction, and parallax not considered.

I. Solution by the PROPOSER.

Let B be the celestial north pole, A the zenith, AB an arc of the meridian equal the co-latitude$=c=50^\circ$, HO a portion of the horizon, SS' and MM'' portions of the diurnal arcs of the Sun and Moon, the Sun setting at S, and the Moon at M'; BS=the polar distance of the Sun$=BS'$, and BM' the polar distance of the Moon, and AM' the zenith distance of the Moon$=90^\circ$.



Produce the vertical circle AM' to S', S' being the place of the Sun when the Moon sets at M'. The line joining the Moon's cusps must be at right angles to the line $M'S'$ joining the centers of the Sun and Moon, and as the horison is at right angles to $AM'S'$, the line of the cusps must lie on the horizon and set when the Moon's center sets. Put $\angle ABS=\phi$=Sun's hour angle when it sets, and $\angle ABS'=\theta$= Sun's hour angle when the Moon sets, and $\angle ABM'=\psi$=Moon's hour angle when it sets.

Then we have $\cos\phi=\tan\delta'\tan\lambda$. $\therefore\ \phi=81^\circ\ 36'\ 29''$, and $\cos\psi=-\tan\delta\tan\lambda$. $\therefore\ \psi=94^\circ\ 32'\ 7''$. Take an auxiliary arc χ', and $\tan\chi'=\cos\psi\cot\delta$. $\therefore\ \chi'=40^\circ\ 0'\ 1''$, then $\cot A=\sin(c-\chi')\cot\psi\,\text{cosec}\chi'$. $\therefore\ A=82^\circ\ 57'\ 55''$. Take an auxiliary angle γ', and $\cot\gamma'=\tan A\sin\lambda$. $\therefore\ \gamma'=10^\circ\ 52'\ 2''$. Then $\cos\gamma'\cot\lambda\tan-\delta'=-\cos\gamma$. $\therefore\ \gamma=101^\circ\ 44'\ 43''$, and $\angle ABS'=\gamma'+\gamma=\theta=112^\circ\ 36'\ 45''$, and $\theta-\phi=31^\circ\ 0'\ 16''=2$ hours, 4 minutes, 1 second.

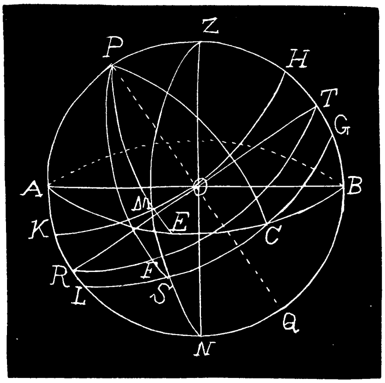
NOTE. The synchronous setting or rising of the cusps of a crescent Moon, is a phenomenon which must occur frequently in the tropics, and rarely or not at all beyond latitude 45°. On the 4th of July, 1897, such a moonset was very nearly accomplished, and another, almost perfect, will occur February 22, 1898, the declinations being then as given in the problem. Few persons in the northern states have ever seen the Moon set with both horns vertical.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let O be the observer, Z his zenith, HMK Moon's path, $GCSL$ Sun's path, $TEFR$ celestial equator, $AMCB$ the horizon. Let M be the position of the Moon when setting. Then, in order that the horns may set at the same time, S, M, where S is the Sun, must be on the same meridian, $ZMSN$.

$AP=\lambda=40^\circ$. $ME=\delta=5^\circ\ 23'$ N. $SF=\delta_1=9^\circ\ 52'$ S. In the triangle

PMZ, $PZ=90^\circ-\lambda_1$, $ZM=90^\circ$, $PM=90^\circ-\delta$. Let $\angle ZPM=h$, $\angle PZM=z$, $ZPC=h_1$, $\angle ZPS=\beta$.



Then $\cos h=-\tan\lambda\tan\delta=-\tan 40^\circ \tan 5^\circ 23'$. $\therefore h=94^\circ\ 32'\ 6.8''$.

$\sin z=\cos\delta\sin h$. $\therefore z=82^\circ\ 57'\ 54''$.

$\cos h_1=-\tan\lambda\tan\delta_1$. $\therefore h_1=81^\circ\ 36^\circ\ 29$.

In triangle ZPS, $PS=90^\circ+\delta_1$, $PZ=90^\circ-\lambda$. $\angle PZS=z$.

$\sin PSZ=(\sin z\cos\lambda)/\cos\delta_1$. $\therefore PSZ=\lambda_1=50^\circ\ 30'\ 22''$.

$$\cot\tfrac{1}{2}\beta=\frac{\sin\frac{1}{2}(180^\circ-\lambda+\delta_1)}{\sin\frac{1}{2}(\lambda+\delta_1)}\tan\tfrac{1}{2}(z-\lambda_1).$$

$\therefore \beta=112^\circ\ 36'\ 47''$.

β=Sun's hour angle at moonset ; h_1= Sun's hour angle at sunset ; $\beta-h_1=31^\circ\ 0'\ 18''$=2 hours, 4 minutes, 1.2 seconds after sunset.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

89. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, N. J.

Solve by pure arithmetic. A criminal having escaped from prison traveled 10 hours before his escape was known ; he was then pursued so as to be gained upon 3 miles an hour ; after his pursuers had traveled 8 hours they met an express going at same rate as themselves, who had met the criminal 2 hours and 24 minutes before ; in what time from the commencement of the pursuit will they overtake him?

90. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

A owes \$6000 which is drawing 6% interest. He wishes to pay off the debt in six equal annual payments, the first to be due in one year. The whole portion of the claim unpaid at the end of each year to be accounted as principal, and to draw interest to the time of the next payment. Required the amount of each payment, so the six equal payments will discharge the obligation, interest and all.

91. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pa.

\$1000.00. Cleveland, Ohio, May 26, 1893.

Two years after date I promise to pay John Davis, or order, one thousand dollars, for value received, interest six per cent. payable annually. J. M. LEWIS.

Indorsements: December 14, 1895, \$560.56; May 11, 1896, \$10.02; June 14, 1697, \$545.06.

Find, by the United States' Rule, the amount due August 2, 1897.

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 1.

GEOMETRY.

86. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, O.**

Prove that the four conics which have S for focus and which touch the three sides of each of the triangles ABC, AEF, BFD, CDE, have their lata-recta equal.

87. **Proposed by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military Academy, Washington, Miss.**

Given any two straight lines in space, AB, CD, which do not intersect. So construct upon one of the lines as base, a triangle, having its vertex in the other line, such that its perimeter shall be a minimum.

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 1.

CALCULUS.

68. **Proposed by EDWARD DRAKE ROE, Jr., A. M., Associate Professor of Mathematics, Oberlin College, Oberlin, Ohio.**

If $a^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}$ to r steps be denoted by $a^{\overset{r}{x}}$, and if $y=x^{\overset{r}{x}}$, prove that

$$D_x y = x^{\overset{r}{x}+\overset{r-1}{x}+\dots+\overset{1}{x}}(\log x)^{r-1}(1+\log x)+\sum_{k=2}^{k=r} x^{\overset{r}{x}+\overset{r-1}{x}+\dots+\overset{k-1-1}{x}}(\log x)^{r-\kappa}.$$

69. **Proposed by GEORGE LILLEY, Ph. D., LL. D., Professor of Mathematics, State University, Eugene, Ore.**

An elliptic fence encloses a field whose major and minor axes are $2a$ and $2b$, respectively. The ends of a rope, the length of which is equal to the length of the fence, are fastened outside the fence and at the extremities of the major axis. A horse is tethered by means of a ring which slides freely on the rope. Over how much ground can he feed? What is the length of its outside border? Find these values in square feet and feet, true to six decimal places, when the area of the field is one acre and $a=2b$.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 1.

MECHANICS.

63. **Proposed by A. H. BELL, Hillsboro, Ill.**

From a horizontal support at a distance of ten feet apart, a beam 5 feet long and 10 pounds weight is suspended by ropes attached to each end. The ropes are 3 and 5 feet respectively, in length. Required the angles made by the ropes and horizontal support. Also the stress upon each rope.

64. **Proposed by FREDERICK R. HONEY, Ph. B., Instructor in Mathematics in Trinity College, New Haven, Conn.**

Let the isosceles triangle abc, whose plane is vertical, and whose base bc is horizontal, and supported at each end b and c, represent three rods jointed at the points a, b, and c. Let any load L be suspended at the vertex a. It is required to find the value of the angle between the sides of the triangle and the base which shall make the sum of the weights of the rods a minimum, the length of the base bc being fixed.

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 1.

DIOPHANTINE ANALYSIS.

60. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.**

It is required to find three positive numbers, such that if each be diminished by five times the fifth power of their sum the three remainders will be rational fifth powers.

61. **Proposed by SYLVESTER ROBBINS, North Branch, New Jersey.**

Investigate that infinite series of prime, integral, rational scalene triangles where the sides of every term are consecutive numbers; then take the necessary factors from the proper KEY, and by an expeditious method, find in their order the areas of ten initial terms.

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 1.

AVERAGE AND PROBABILITY.

59. **Proposed by J. SCHEFFER, A. M., Hagerstown, Md.**

A circle is rolling along a horizontal straight line. The uniform velocity of the center is v. Find the average velocity of a point of the circumference.

60. **Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

Four points are taken at random within an ellipse. What is the chance that they form a reentrant quadrilateral.

*** Solutions of these problems should be sent to B. F. Finkel, not later than March 1.

MISCELLANEOUS.

58. **Proposed by EDMUND FISH, Hillsboro, Ill.**

The longest noonday winter shadow of an upright object is found to be seven times as long as the shortest summer shadow of the same object. Required the latitude of the place.

59. **Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.**

When a cylindrical china jar, standing upon the ground, receives the sun's rays obliquely, a bright curve is observed to form itself at the bottom of the jar, and it is found that the shape and dimensions of this curve are not affected by the varying elevations of the sun: account for this latter circumstance, and determine the nature of the bright curve. [From *Parkinson's Optics.*]

*** Solutions of these problems should be sent to J. M. Colaw, not later than March 1.

EDITORIALS.

Contributors desiring to have their portraits appear in the next group should send us a good photograph from which the plate is to be made.

We will pay twenty-five cents each for a limited number of Nos. 6 and 11, Vol. II. of the MONTHLY, also fifteen cents for a copy of the October *Cosmopolitan Magazine*.

This number of the MONTHLY is sent to all our old subscribers. Those wishing to discontinue should return this number with their name and address written on the wrapper.

In the February number of the MONTHLY will appear a biography of Bolyai János, by Dr. Halsted. nearly all of which is absolutely new and never before published. Also in addition to Dr. Lovett's valuable article, will appear a New Solution of the Cubic Equation, by Dr. L. E. Dickson.

Dr. Wm. B. Smith, of the Tulane University of Louisiana, informs us that his *Infinitesimal Analysis, Vol. I.* will appear in about two months. Judging from the character of Dr. Smith's previous works, viz., *Introductory to Modern Geometry and Coördinate Geometry*, this work will prove a valuable addition to the literature of the subject.

Prof. W. W. Beman, of the University of Michigan, calls our attention to the fact π was first used to represent the number 3.141592 . . . in Jones' *Synopsis Palmariorum*, London, 1706. This fact seems to have been overlooked by most writers on mathematics as we find the statement that Euler was apparently the first to use the symbol π, *Britannica Encyclopedia*, ninth edition.

The reorganization of the Mathematical Society of Utah took place December 28, 1897, at Provo, electing the following officers : President, W. J. Kerr, President Brigham Young College, Logan, Utah ; Vice President, D. H. Adams, Principal Madison School, Ogden ; Treasurer, G. N. McKay, Principal Lowell School, Salt Lake City ; Secretary, L. M. Gillean, Teacher of Mathematics, Salt Lake High School. The object of the Society is to promote the interest of mathematics in general in the State.

The following letter has been sent us :
To the Editor of THE AMERICAN MATHEMATICAL MONTHLY.

PROPOSED SYLVESTER MEMORIAL.

Sir : May I be permitted to appeal through your columns to all friends and admirers of the late Professor J. J. Sylvester to assist in founding a suitable memoriol in honor to his name and for the encouragement of mathematical science. A movement was inaugurated on this side of the Atlantic soon after his death and it was resolved by the promoters that a fund should be raised for the purpose of establishing a Sylvester Medal to be awarded at certain intervals for mathematical research to any worker irrespective of nationality. For the purpose of carrying out the scheme a strongly representative Internationel Committee has been formed and I should like to take advantage of this opportunity of expressing the great satisfaction which it has given to the promoters to be enabled to include in this Committee so many great and distinguished names from the Amer-

ican Universities. In every case our invitation to join the Committee has been most cordially responded to and the consent has in many instances been accompanied by expressions of the greatest sympathy and encouragement. The list as it stands practically includes the leading mathematicians of the whole world.

It has been estimated that a capital sum of 5000 dollars will be sufficient for the proposed endowment and of this about one-half has already been subscribed here. In appealing to the American public to enable us to complete the desired sum I am in the first place prompted by the consideration that Sylvester's association with the Johns Hopkins University and the leading part which he took in advancing mathematical science in America renders his claim to estimation on the part of the citizens of your country quite a special one. It is but a modest endowment that we are asking for and I am sure that all those who were personally acquainted with him and who realize the great influence which he exerted in raising the intellectual level of every Institution with which he was associated will be glad of this opportunity of coöperating in the movement.

It is proposed that the fund when complete shall be transferred to the Council of the Royal Society of London, that body having undertaken to accept the trust and to award the medal triennially to mathematicians of all countries. I can hardly venture to trespass upon your courtesy to the extent of asking you to print the complete list of our Committee, but for your own information I beg to send a copy herewith. It will be sufficient to state that it comprises the names of President Gilman of the Johns Hopkins University, of Professor Simon Newcomb of Washington, of Professor Willard Gibbs of Yale, of Professor Peirce of Harvard, and many other well known American men of Science. Subscriptions may be sent to and will be acknowledged by Dr. Cyrus Adler, the Smithsonian Institution, Washington, or by Dr. George Bruce Halsted, President of the Academy of Science, 2407 Guadalupe Street, Austin, Texas.

I am, Sir, Yours obediently,

RAPHAEL MELDOLA,

Professor in the Finsbury Technical College, London, England.

Hon. organizing Secretary to the Sylvester Memorial.

December 1897.

BOOKS AND PERIODICALS.

School Algebra Complete. By Fletcher Durell, Ph. D., and Edward R. Robbins, A. B., Mathematical Masters in the Lawrenceville School. 392 pages. 1898. Harrisburg, Pa.: R. L. Myers & Co.

The effort of the authors has been to simplify principles and make them attractive, by showing as plainly as possible the practical reason for each step or process. The theory of the subject is developed by showing that new symbols are introduced into algebra for the sake of definite advantages in representing numbers, and that the fundamental laws governing their use derive their importance from the economies which they make possible

in dealing with the symbols for numbers. It is here shown that each successive step is taken up for the sake of the new power which it gives as compared with previous processes. While following this line in the development of the subject, no radical departures are made in this book, both the order of topics and the treatment in other respects following closely the best practice as presented in the text-books most used at present.

The selection of examples seems to have been made with excellent judgement and especial care has been exercised in their gradation. The completeness of the book is aided by the addition of valuable short chapters on Permutations and Combinations, Undermined Coefficients, The Binomial Theorem, Continued Fractions, and Logarithms. The treatment in this work is clear and satisfactory, and especially so in the important subjects of factoring, fractions, exponents, and radicals. The authors are practical and successful teachers, and their book contains a very fair presentation of the latest and best methods of treating the subject. J. M. C.

The Equations of Hydrodinamics in a form suitable for application to Problems connected with the Movements of the Earth's Atmosphere. Prepared at the request of Willis L. Moore, Chief of Weather Bureau. By Joseph Collier, Columbia University. Published by authority of the Secretary of Agriculture. Folio Pamphlet, 8 pages.

A New Astronomy for Beginners. By David P. Todd, M. A., Ph. D., Professor of Astronomy and Director of the Observatory, Amherst College. 12mo. Cloth, 480 pages, with colored plates and copious illustrations. Price, $1.30. New York, Cincinnati, and Chicago : American Book Co.

This is by far the most interesting and attractive elementary astronomical work we have seen. Everything in the book is of high scientific and educational value. The illustrations are very artistic and so developed as to give the student an impressive notion of the objects illustrated. We very highly commend this work to all who are looking for a first-class elementary astronomy. B. F. F.

On the Commutator Groups. By Dr. G. A. Miller. Reprinted from the Bulletin of the American Mathematical Society, being a paper read before the Society at its Fourth Summer Meeting, Toronto, Canada, August 17, 1897.

An Algebraic Arithmetic, being an Exposition of the Theory and Practice of Advanced Arithmetic based on the Algebraic Equation. By S. E. Coleman, B. S., William Whitney Fellow at Harvard University, formerly Instructor in Mathematics in the Oakland High School, Oakland, Cal. 8vo. Cloth, 151 pages. Price, 50 cents. New York : The Macmillan Co.

This is another one added to a number of arithmetics recently published which if extensively used in the publish schools will go far towards breaking down the barriers between arithmetic and algebra. This book and others of its kind will be considered by ye pedagogue of ye olden times as an icónoclast. But let it be; from the broken images will rise better results for the mathematics of the future. Of course, it is probable that pupils will always have to commit to memory the multiplication table and to learn other facts about numbers, but when these facts are well fixed in the mind, why continue to manipuiate figures as the symbols of numbers when other and simpler characters may be used to represent a number which is represented by a great many figures? To economize time, to facilitate computations, and to secure better results in teaching elementary mathematics is precisely the purpose of this book. It often makes use of letters to represent number, and introduces the equation from the first. B. F. F.

The Annals of Mathematics. Edited by Wm. H. Echols. Published under the auspices of the University of Virginia. Bi-Monthly, price \$2.00 per year in advance.

The October (1897) number of the *Annals of Mathematics* contains the following articles: The Analytical Representation on a Power of Prime Number of Letters with a Discussion of the Linera Group, by Dr. L. E. Dickson; Note on Integral and Integro-Geometrico Series, by Prof. Edward Drake Roe; Note upon a Representation in Space of the Ellipses Drawn by an Ellipsograph, by Prof. E. M. Blake. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, \$1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson.

The principal articles of the February number are: The Selection of One's Life Work, by E. Benjamin Andrews; The Great Electric Trust, by Francis Lynde; and Personnel of The Supreme Court, by Nannie-Bille Maury.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, \$2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

Cuba, Hawaii, and China furnish the principal topics discussed editorially in the *American Monthly Review of Reviews* for February. There are also a few paragraphs of pointed comment on current domestic politics—the factional differences between Ohio Republicans and the swelling tide of Crokerism in the Democratic party. The editor gives his views on Tammany's attitude toward the New York rapid-transit problem and on the reckless expenditure of canal-improvement funds by the Republican bosses of the State.

Some Errata in December Number.

Page 316, line 14, for "$[ac=bd]$" read $[ac+bd]$.
Page 317, lines 20 and 21, in all denominators, for "x^2-y^2" read x^2-yz.
Page 317, line 21, in numerator, for "$[x^2+y^2]$" read $[x^2+yz]$.
Page 317, line 22, for "$[x^2-y^2]$" read $[x^2-yz]$.
Page 318, line 4, for "(1), (5) and (6)" read (1), (10) and (11).
Page 320, Fig. 1 should be reversed in position to correspond to Figs. 2 and 3.
Page 320, line 15, for "$+h\cot[\theta+\alpha]$" read $-h\cot[\theta+\alpha]$.
Page 320, line 18, where "x/z" occurs read $x/2$.
Page 320, line 21, read $[\frac{1}{3}\pi h][x/2]z$.
Page 321, line 14, read $[b-a]^2$ in the first denominator.
Page 322, line 12, for "$\angle DAC$" read $\angle DAB$.
Page 322, line 14, insert) at end of line.
Page 322, in Fig. 3, E and H should be interchanged.
Page 322, line 22, for "$\left(1+\frac{s\sqrt{\beta^2+1}}{\alpha\beta}\right)$" read $\left(1-\frac{s\sqrt{\beta^2+1}}{\alpha\beta}\right)$.
Page 323, line 2, for "$[+\beta]$" read $[k+\beta]$.
Page 323, line 5, insert $\}$ before $=$.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. FEBRUARY, 1898. No. 2.

BIOGRAPHY.

BOLYAI JÁNOS. [JOHN BOLYAI.]

BY GEORGE BRUCE HALSTED.

FOR human thought, for culture, for intellectual freedom, for philosophy it is impossible to overestimate the tremendous importance of the non-Euclidean Geometry. The life of Bolyai János will therefore possess imperishable interest for mankind.

He was born December 15th, 1802, at Kolozsvár. His mother, Benkö Zsuzsánna, gifted, beautiful, imaginative, was devoted to her son. His father, Bolyai Farkas, whose biography is given in Vol. III., No. 1., of this MONTHLY, declared the son a marvel, a wonder-child.

Yet until long after both father and son had died, not one published word, except from the father, ever called the slightest attention to an achievement of genius by the son so far-reaching, so profound as to modify all present and future thought.

Yet the neglected János had the just appreciation given to genius of its own work. Speaking once of his own early youth he says : "The fore-knowledge was withheld from me that I should perfect the theory of mathematics and in this way bless the human race." In the great chest at Maros-Vásárhely wherein his papers are preserved I found an autobiographic fragment whick speaks of him as "the phœnix of Euclid." Yet no picture of him is known to exist, and I deemed it a discovery of interest to find from his passport that his eyes were blue.

From childhood great pains were taken with his education. The brigthest of his father's students taught him, except in mathematics, where the father himself was his teacher. His progress in mathematics was so lightning-quick, that, as his father often related, he waited not for the proof of the theorem or the solution of the problem, but gave them himself. ''Like a demon sprang he on before me,'' says the father, ''and bade me go further.''

At twelve years of age he passed the ''Rigorosum'' covering six gymnasial classes, and according to the usage of that time became a ''student.'' At this examination his translations from Magyar into Latin were written in the style of Tacitus, gaining the professor's unstinted praise.

János was two years a ''student,'' or rather he for two years attended the college to play checkers. He was rarely present at the lectures. At the approach of the winter examinations, the professor of history, John Antal, later bishop of the reformed church, complained of him to his father. The father called up the idler and gave him some good advice, which he did not take. He went up to the examinetion after glancing through the lectures, and answered beautifully. Whatever and however much he was asked, he was always fully prepared.

Just so in the next semester he carried on his checker-playing ; but now the professor said nothing, and again the examination was a brilliant success.

When the father was sick he sent his son to teach the college classes in his stead ; and the students preferred the teaching of the thirteen-year-old János to that of the father—they got more out of it. Even in his twelfth year he played the violin so well, that he was able to execute the most difficult pieces at sight.

At that time was given perhaps the first opera in Maros-Vásárhely. At the presentation the blind composer was there. A young Saxon played first violin, János second. During the representation a string broke on the violin of the prima, the Saxon. They exchanged the notes, and the child János played at sight first violin. The blind composer, who had hitherto been dissatisfied, cried out, ''Bravo ! now the prima dominates !''

At the age of fifteen János went away to the great Engineering Academy in Austria. Not long after, the Archduke John visited the Academy as inspector, and asked that some of the new students be called upon. When the choice fell on young Bolyai, he quickly worked out the problem given by the Professor, and went on to the next, and so on. The Archduke was astonished at the genius of the youth, interrupted him with praises, and said to the Professor, ''The other students should be put under this one ; he knows more than the whole class.''

After this, Magnates came from Erdély came to Vienna. At an audience with the Archduke he told them what a perfect genius of a boy from Erdély was at the Academy. He asked if they knew his father. They answered, the father was also a genius and their friend. The Archduke sent him word that he was delighted with his son, who if he would behave himself might count on a great career. János finished the five years course with the highest distinction, and at twenty entered as Cadet the Engineering Corps. At twenty-one he was Lieuten-

ant. In the whole army he was the first as mathematician, first as violinist, but also first with his sabre. He fought duels continually, several with fatal ending, he always victor. Once he was challenged by thirteen officers of a cavalry regiment. He accepted all with the condition that he was to be allowed to play a piece on his violin after every two duels. As always he was victor in all.

His conduct was haughty. But remember, in less than three months from his graduation he had, as he says in his letter of November 3d, 1823, "*from nothing created another wholly new world.*" He had solved the problem of the ages, the problem of parallels, and had made a new kind of universal space.

In August, 1823, he was graduated at the royal Engineering Academy as Underlieutenant, and ordered for service to Temesvár. He left Vienna on September 17th and arrived at Temesvár on September 30th. His extraordinary creation falls thus in the very first month of his service, and before his twenty-first birthday.

The splendid youth, who knew that he had succeeded where the very greatest of his predecessors on earth had failed, could scarcely have been expected to submit to any affront. His sabre was ready for whoever wished to impale himself.

Franz Schmidt has preserved a glimpse of János at Temesvár in the story often told him by his father, Anton Schmidt, of a young officers of engineers with whom he feared to come in contact, who to prove the might of his arm and the temper of his Damascus blade, was accustomed to show his visitors how with one blow he could cut off a heavy nail driven into his door-post. But not even this trenchant blade could open any brain about him to appreciation of his immortal achievement.

Only his father knew and publicly declared what he had done.

János has left on record that it was in the Engineering Academy that he began to investigate the properties of the line everywhere equidistant from a straight line, on the hypothesis that it was not straight, and on that investigation corresponded with his father in 1820. But not until 1823 did he completely penetrate and master the whole subject. He wrote out a treatise on the subject, which in 1825 he transmitted to his former professor Johann Wolter von Eckwehr.

His father Bolyai Farkas induced him to translate this into Latin and issue it as an Appendix to the first volume of the father's "Tentamen." It was printed separately and appeared before the volume with which it was afterward bound up.

After years, 1843, in the second edition of his Arithmetic (in Magyar) the father says: "The *Appendix* is a work worth whole folios. It is to the true geometer a work beautiful, necessary, original, colossal. How many thinkers have even up to the latest time endeavored in vain to make sure the one foundation of the edifice of Euclidean geometry? In the mentioned work is built up an absolute and for all cases true geometry. Few have appreciated its worth, yet it is in one word a true classic."

After more than half a century the world has come to that opinion. Says F. S. Macauley in 1897, "I was simply fascinated with your translation of Bolyai's 'Science Absolute' when I first read it, as indeed almost any one would be who comes across it."

A NEW SOLUTION OF THE CUBIC EQUATION.

By DR. L. E. DICKSON.

The method here suggested is analogous to that used by Ferrari in solving the biquadratic. I multiply the reduced cubic by $x-c$ and determine c so that the resolvent cubic shall be binomial. Thus, adding $(ax+b)^2$ to each member of

$$(x-c)(x^3+px+q)=0$$

and requiring the new equation to take the form

$$(1)\qquad [x^2-(c/2)x+\lambda+(p/6)]^2=(ax+b)^2$$

we have the conditions on a, b, λ :

$$(2)\qquad \begin{aligned} a^2&=2\lambda+\tfrac{1}{4}c^2-\tfrac{2}{3}p \\ 2ab&=-c\lambda+\tfrac{5}{6}cp-q \\ b^2&=(\lambda+\tfrac{1}{6}p)^2+qc. \end{aligned}$$

Since c is a root of (1), we may set

$$(3)\qquad \tfrac{1}{2}c^2+\lambda+\tfrac{1}{6}p=ac+b,$$

limiting ourselves to one of the two sets of values for a and b in terms of λ and c. Equating the two values for $4a^2b^2$ given by (2), we reach the resolvent cubic for λ in *reduced* form :

$$(4)\quad 8\lambda^3+2\lambda(3cq+c^2p-\tfrac{1}{3}p^2)+(qc^3-cpq-q^2-\tfrac{2}{3}p^2c^2-\tfrac{2}{27}p^3)=0.$$

Its discriminant is found to be

$$R(c^3+pc+q)^2,\text{ where } R\equiv\tfrac{1}{4}q^2+\tfrac{1}{27}p^3$$

is the discriminant of the given cubic.

The coefficient of 2λ in (4) vanishes for

$$(5)\qquad \tfrac{1}{3}pc=-\tfrac{1}{2}q+\sqrt{R},$$

whence
$$c^3+pc+q=\frac{108\,R}{p^3}(-\tfrac{1}{2}q+\sqrt{R}).$$

Hence, if we determine c by (5) the cubic (4) becomes

$$(6)\qquad 8\lambda^3=-q(c^3+pc+q)+8R=\left(\frac{6\sqrt{R}}{p}\right)^3(-\tfrac{1}{2}q+\sqrt{R}).$$

Then by (2), $a^2=\frac{6\sqrt{R}}{p}(-\frac{1}{2}q+\sqrt{R})+\frac{1}{4}c^2-\frac{2}{3}\gamma$.

By a slight calculation, we find

$$a=\frac{1}{2}c+\sqrt[3]{-\frac{1}{2}q+\sqrt{R}}+\sqrt[3]{-\frac{1}{2}q-\sqrt{R}}.$$

This would be expected, since by (3)

$$(x-c)(x-a+\tfrac{1}{2}c)=x^2-\tfrac{1}{2}cx+\lambda+\tfrac{1}{6}p-ax-b,$$

whence $a-\frac{1}{2}c$ is a root of the given cubic.

University of California, January 17.

THE FIRST AWARD OF THE LOBACHEVSKI PRIZE.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society ; and Professor of Mathematics in the University of Texas, Austin, Texas.

The Lobachévski prize is adjudged every three years. Its value is five hundred roubles. It is given for work in geometry, preferably non-Euclidean geometry. All works published within the six years preceding the award of the prize and sent by their authors to the Physico-Mathematical Society of Kazan are allowed to compete if published in Russian, French, German, English, Italian, or Latin.

The Society has now in formal session awarded the prize to Sophus Lie, Professor of Mathematics at the University of Leipzig, for his work "Theorie der Transformationsgruppen. Band III. Leipzig, 1893." In this work the theory of non-Euclidean geometry has been exhaustively re-stated and re-established in a profound investigation of the work of Helmholtz on the space-problem.

To the genius of Helmholtz is due the conception of studying the essential characteristics of a space by a consideration of the movements possible therein.

But since the time when Helmholtz did his work on this subject, the greatest of living mathematicians, Sophus Lie, formerly of Christiana, has enriched mathematics with a new instrument, the Theory of Groups, which its creator has applied with tremendous power to the Hemholtz treatment. Lie finds, as was almost inevitable, that certain details had escaped the great physicist, but that, with the tact of true genius, he had kept his main results free from error, though there comes to light a superfluity in his explicit assumptions, an unconscious assumption now seen to be mathematically important for the rigor of the demonstration, and at least one definits error in minor results.

Lie's method is in general the following. Consider a tridimensional space, in which a point is defined by three quantities, x, y, z.

A movement is defined by three equations :

$$x'=f(x, y, z) \; ; \quad y'=\varphi(x, y, z) \; ; \quad z'=\psi(x, y, z).$$

By this transformation an assemblage A of points (x, y, z) becomes an assemblage A' of points (x', y', z').

This represents a movement which changes A to A'. Now make, in regard to the space to be studied, the following assumptions :

1st. Assume : in reference to any pair of points which are moved, there is *something* which is left unchanged by the motion. That is, after an assemblage of points A has been turned by a single motion into an assemblage of points A', there is a certain function F of the coördinates of any pair of the old points (x_1, y_1, z_1), (x_2, y_2, z_2) which equals that same function F of the corresponding new coördinates (x_1', y_1', z_1'), (x_2', y_2', z_2') ; that is $F(x_1, y_1, z_1, x_2, y_2, z_2) = F(x_1', y_1', z_1', x_2', y_2', z_2')$. This *something* corresponds to the Cayley definition of the distance of two points when interpreted as completely independent of ordinary measurement by superposition of an unchanging sect as unit for length.

This independence, involving the determination of cross-ratio without any use of ordinary ratio, without using congruence, without using motion, Cayley never clearly saw. It follows from the profound pure projective geometry of von Staudt.

2nd. Assume : if one point of an assemblage is fixed, every other point of this assemblage, *without any exception*, describes a surface (a two-dimensional aggregate).

When two points are fixed a point in general (exceptions being possible) describes a curve (a one-dimensional aggregate). Finally, if three arbitrary points are fixed, all are fixed (exceptions being possible). With these assumptions Lie proves exhaustively that the general results of Helmholtz and Riemann follow : that is, there are three and only three spaces which fulfill these requirements, namely, the traditional or Euclidean space, and the spaces in which the group of movements possible is the projective group transforming into itself one or the other of the surfaces of the second degree $x^2+y^2+z^2\pm 1=0$.

In the appreciation of this work of Lie's prepared for the Society by Felix Klein, for which the Lobachévski gold medal was given him, he says that Lie's work stands out so prominently over all the others to be compared with it that a doubt as to the award of the prize would scarcely have been possible. Decisive for this judgment as to the height of the scientific achievement is not only the extraordinary depth and keenness with which Lie in the fifth section of his book handles what he has called the Riemann-Helmholtz space problem, but especially the circumstance that this treatment appears so to say as logical consequence of Lie's long-continued creative work in the province of geometry, especially his theory of continuous transformation-groups.

The extraordinary importance which the works of Lie possess for the general development of geometry can scarcely be over-estimated. In the coming years they will be still more widely prized than hitherto. Passing then to the consideration of the present state of the space question, Klein takes up the origin

of axioms. Whence come the axioms ? A mathematician who knows the non-Euclidean theories would scarcely maintain the position of earlier times that the axioms as to their concrete content are necessities of the inner intuition.

What to the uninitiated appears as such necessity, shows itself after long occupation with the non-Euclidean problems as result of very complex processes, and especially education and habit.

Do the axioms come from experience ? Helmholtz energetically says yes ! as is well known. But his expositions seem in a definite direction incomplete. One will, in thinking over these, willingly admit that experience plays an important part in the formation of axioms, but will notice that just the point especially interesting to the mathematician remains untouched by Helmholtz.

It is the question of a process which we always complete in exactly the same way in the theoretical handling of any empirical data, and which therefore may seem quite clear to the scientist.

Expressed generally : *always the results of any observations hold good only within definite limits of precision and under particular conditions ; when we set up the axioms, we put in the place of these results statements of absolute precision and generality.*

In this "idealizing" of empirical data lies in my opinion the peculiar essence of axioms. Therein our addition is limited in its arbitrariness at first only by this, that it must cling to the results of experience and on the other hand introduce no logical contradiction.

Then enters as regulator also that which Mach calls the "economy of thinking." No one will rationally hold fast to a more complicated system of axioms when he sees, that with a simpler system he already completely attains the exactitude requisite to the representation of the empirical data.

Klein goes on to mention the possibility of a series of topologically distinguishable space-forms built of limited (simply compendent) space-pieces either all Euclidean, all Lobachévskian, or all Riemannian. Beside these three just mentioned family-types, the parabolic, the hyperbolic, the single elliptic, Killing has shown that the spherical, in which two geodetics always cut in two points, is the only one which as a whole is freely movable in itself.

Then Klein says : I consider all the topologically distinguished space-forms as equally compatible with experience. That in our theoretic considerations we prefer some of these space-forms (namely the family types, that is the properly parabolic, hyperbolic, elliptic) in order to finally assume the parabolic geometry, that is the customary Euclidean geometry as valid, happens simply from the principle of economy.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

78. Proposed by **J. MARCUS BOORMAN**, Consultative Mechanician and Counselor at Law, Woodmere, Long Island, N. Y.

Solve $x^2+xy=10......(1)$; $y^2+xy=15......(2)$, for all the roots, and prove that they are the roots.

I. Summary of Solutions by J. OWEN MAHONEY, M. Sc., Lynnville, Tenn.; F. R. HONEY, Ph. B., Instructor at Trinity College, New Haven, Conn.; J. SCHEFFER, A. M., Hagerstown, Md.; G. B. M. ZERR, A. M., Ph. D., The Russell College, Lebanon, Va.; P. S. BERG, A. M., Principal of Schools, Larimore, N. D.; A. H. BELL, Hillsboro, Ill.; H. C. WILKES, Skull Run, W. Va.; CHARLES C. CROSS, Laytonsville, Md.; I. H. BRYANT, A. M., Ft. Smith High School, Ft. Smith, Ark.; and COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.

Divide (1) by (2), then $x/y=\frac{2}{3}.....$ (3) ; x and y must be plus or minus together. Then x from (3) in (1), gives $y^2=9$, or $y=\pm3$. Also, $x(=2y/3)=\pm2$. $\therefore$ $x=\pm2$, $y=\pm3$ are the roots, which can be proved by direct substitution. [MAHONEY, HONEY, SCHEFFER, AND BELL.]

Adding (1) and (2), and extracting square root, $x+y=\pm5......(4)$. Then (3) in (4) gives, $y=\pm3$, $x=\pm2$. [ZERR.]

Subtracting (2) from (1), $x^2-y^2=-5......(5)$. Then (5) by (4), $x-y=\mp1......(6)$. From (4) and (6), $x=\pm2$, $y=\pm3$. [BERG, AND WILKES.]

Solve (1) for x, then $x=[-y\pm\sqrt{(40+y^2)}]/2$ $....(7)$. Substituting (7) in (2), and reducing, $y=\pm3$. Similarly, $x=\pm2$. [CROSS.]

Substitute vy for x and solve ; then $v=\frac{2}{3}$ or -1. By substituting and reducing, $x=\pm2$, and $y=\pm3$. Then if $v=\frac{2}{3}$, $x=\pm2$, $y=\pm3$. If $v=-1$, $x^2=\infty$, $x=\pm\infty$, $y=\mp\infty$. The first values satisfy the equations. Substitute the second values, $x^2=\infty$, $x=+\infty$, $y=-\infty$, $xy=-\infty$, $x^2+xy=\infty-\infty$. Since $\infty-\infty$ is an indeterminate expression, it may equal any numbers. Therefore the equations are satisfied for the last values of x and y as well as for the first values. [BRYANT.]

From (1), $y=(10-x^2)/x$. In (2), we get $10-x^2+(100-20x^2+x^4)/x^2=15$. Whence $25x^2-100=0......(8)$; or $x=\pm2$. Being an equation of the fourth degree we ought to have four answers. We can write (8), $0x^4+0x^3+25x^2-100=0$. Since coefficients of two highest powers are zero, this indicates two infinity roots, which may be *claimed* defective in solutions in print. [SCHMITT.]

See *Analyst*, Vol. VIII, page 111, and Vol. IX, page 53, as well as solution below, for discussion as to whether $x=\pm\infty$ and $y=\mp\infty$ are also roots of the equations. [EDITOR.]

II. Solution by the PROPOSER.

The equations are fourth degree in x ; y ; the *singular* case in $x^2+xy=a$ $.....(A)$; $y^2+exy=A+d.....$ (B). *sub-ultimate* fourth degree when the positive are the negative roots reversed in signs. Here $a=10$; $A=15$; e (*variable*) $=1$; $d=F(e-1)=0$.

Transpose (I), $x^2=10-xy$; then transpose (II), and multiply, giving $x^2y^2=150-25xy+x^2y^2$......(III). $\therefore$ $xy=6$......(IV), and $(1-1)x^2y^2-25xy+(12\frac{1}{2})^2[1/(1-1)]=(12\frac{1}{2})^2/(1-1)-150$......(III$a$). By (IV) and (I), (II); $x^2=4$; $y^2=9$; $\therefore$ $x=\pm2$; to $y=\pm3$; four *true* roots......(V). But x^2y^2 (vanished) is quadratic; it has therefore a *second* root $-xy=-6$; hence in (I), $x^2-(-6)=10$; *i. e.* $x^2=10+(-6)$......(VI). $\therefore$ extract negatively $x=\mp2$; $y=\mp3$ [by parity in (II)]; and $x=\pm2$; $y=\pm3$; ......(V) are the eight required roots of Case (I), (II) and are *all* its roots. Q. V. D.

Presumably (*visum*) demonstrated.

Proof. Case (I) (II) is *biquadrate* because $xy=6$ that yields *two* pairs of roots flows *equally from* both only possible extractions of the *quadrate* x^2y^2...... $\therefore$ $-xy=6$ *must* yield, failing other derivatives of (III), also the second set of roots. For now obviously (VI) cannot be $x^2=16$ to $y^2=21$ (nor $y^2=9$); because then (I) a whole number $16+4\surd(21)=10$; or an irrational surd=a rational number, *i. e.* reducing $\surd(21)=\mp\frac{3}{2}$, which is *not* true. Nor can $x=\int\surd(-1)$; if so therefore $y=\int_{|}[-\surd(-1)]$; to make xy positive). $\therefore$ $xy-\int^2 \mathrm{I}=10$.... is (I); $xy-\int^2_{|}\mathrm{I}=15$ is (II) which *cannot* be for 10; 15; both positive. If it can be then ..... (I); (II) become: (I) $xy-x^2=10$ and (II) $xy-y^2=15$; then must $\mp_{|}y$; to $\pm_{|}x$; be true. Deduct (I) from (II), $\therefore$ $x^2-y^2=5=st$....(VIII). Let $x+y=s$; $\therefore$ $x-y=t$; add, $\therefore$ $x=\frac{1}{2}(s+t)$; $y=\frac{1}{2}(s-t)$; $\therefore$ $xy=\frac{1}{4}(s^2-t^2)$. Multiply by 4, change signs, etc., $\therefore$ $x^2=\frac{1}{4}(s^2+2st+t^2)$ put into (I), $s^2+2st+t^2-4xy=-40$ ....(IX), [$\pm$ are changed by $\surd(-1)$]; and $y^2=\frac{1}{4}(s^2-2st-t^2)$ put into (II), $s^2-2st+t^2-4xy=-60$. Add the 2 first, $\therefore$ $x^2+y^2=\frac{1}{2}(s^2+t^2)$, replace in (I); then $2x^2-4xy+2y^2+2st=-40$; but (VIII), $st=5$; $\therefore$ $(x-y)^2=-20-5=-25$; *i. e.* $x-y=\pm5\surd(-1)$......(X), and by (VIII) $x+y=\mp\surd(-1)$ .... (XI); add (X), (XI), $\therefore$ $2x=\pm4\surd(-1)$; $x=\pm2\surd(-1)$; and by (XI)$-$(X) $y=\mp3\surd(-1)$; *which do not* satisfy (I), (II). $\therefore$ x; y are not imaginary roots of (I) (II) and equation (IX) is falsely put as equation (I). But correct the false factors $\surd(-1)$ in (IX), (X), (XI) and they by same process yield the roots of (I) (II) found.

Second (see note).

(I) (II) have *no sort* of unreal roots. If they may $x={}_{|}a+ei$; $y={}_{|}b+ci$; $\therefore$ $x^2=a^2-e^2+2_{|}aei$ and $y^2=b^2-c^2+2_{|}bci$. $\therefore$ $ab-ec+(ac+be)i=xy$....(N), and $b^2-a^2+e^2-c^2=5=$(II)$-$(I)......(P); because to cancel i (I) (II) we *must have*, $2aei=(ac+be)i=2bci$......(Q). For *else* rationals 10; 15; (a^2-e^2), etc., *have to* equal $f\surd(-1)$ or $f_{|}\surd(-1)$; etc., *i. e.* real numbers *can be* partly *un*-real, which is absurd! $\therefore$ by (Q) $2a=b+ac/e$; and $2b=a+be/c$...... [These easily reduce to $c=\int i$; $e=\int_{|} i$...... multiplied by i. $\therefore$ $ci=fii$; a *real* number. So our postulate that *ci can be un*-real is false.] $\therefore$ $2b-a=be/c$ and $2a-b=ac/e$. $\therefore$ $b(2-e^2/ec)=a$ and $a(2-c^2/ec)=b$. [Thence $b<a$ and $b>a$! if $c=e$, as here

proven.] Multiply $\therefore ab[4-2(c^2+e^2)/ec+1]=ab$; cancel $ab=ab$, reduce, etc., $\therefore 2ce=c^2+e^2$, $\therefore (c-e)^2=0$......(K). $\therefore c=e$; ...... $\therefore (P_{|})$ is $b^2-a^2=5$ and our assumed *un*-real ci ; ei ; *do not exist.* $\therefore x=a_{|}$; $y=b_{|}$ *real* numbers in (N) (Q) just as we found above.

Again, (IIIa) reduced is $\surd(1-1)xy=[\pm 1/\surd(1-1)]\{[12\frac{1}{2}\pm\surd[156\frac{1}{4}-150(1-1)]\}$......(XII). $\therefore xy=[\pm 1/(1-1)](12\frac{1}{2}\pm 2\frac{1}{2})$. $\therefore x_2y_2=\pm 15/(1-1)$; $x_ay_a=\pm 10/(1-1)$......(XIII). [Unless in $\int 1/(1-1)$ of (XII); $-150(1-1)=0$, for which value $xy=[1/(1-1)](12\frac{1}{2}\pm 12\frac{1}{2})=x_3y_3=25/(1-1)$, or $x_4y_4=12\frac{1}{2}[(1-1)/(1-1)$......(XIV)]. Hence by (XII) and (I), (II) $x_a{}^2=\pm 10[1-1/(1-1)]$; $y_a=\pm 10[\frac{3}{2}-1/(1-1)]$; $x_2^2=\pm 10[1-1.5/(1-1)]$; $y_2^2=\pm 15[1-1/(1-1)]$; multiply x_2 ; y_2 ; $\therefore x_2y_2=\surd\{15[10-10/(1-1)]\}$; (or ?) .....(XV) ; but (XV) *is not* equation (XIII) nor does it nor (XIV) any way satisfy (I) (II). $\therefore x_a, y_a, x_2, y_2$, are *not*, either as by (XIII) or (XIV) roots of (I) (II). Q. E. D.

Last, put $r=$ ratio $y:x$. $\therefore y=rx$; $x^2(1+r)=10$......(Ia) ; $x^2(r^2+r)=15$ .....(H). $\therefore r=\frac{3}{2}$; $r_{|}=-1$......(K). Ratio $\frac{3}{2}$ gives $x=\pm 2$ to $y=\pm 3$..... (V), or $x=\mp 2$; $y=\mp 3$......(VI) above. But by $r_{|}=-1$; (Ia) gives $x^2(1-1)=10$; $x_5=\pm_{|}\surd[10/(1-1)]$; and (H) $x_0=\pm_{|}\surd[15/(1-1)]$; $\therefore y_5=\pm_{|}\surd[10/(1-1)]$; $y_0=\mp_{|}\surd[15/(1-1)]$; ......(M). Whether or no (M) be the quasi roots of (XIV) *none* satisfy (I) (II)! Besides they are *too many* and give (I)(II) 10, 14 or 18 roots! So quasi-results (XV) (M) *are not* roots and ratio $r_{|}=-1$ yields *no* root. $\therefore$ ratio $\frac{3}{2}$ covers all eight roots of (I) (II), viz. roots (V) direct, or (V) with contrary signs as above. Q. V. D.

Finally, solve (I) (II) *generally*. $\therefore x=\pm a/\surd(a+A)$; $y=\pm A/\surd(a+A)$; $xy=aA/(A+a)$. Now $d=(e-1)[(aA)/(a+A)]$, so equation (A) is equation (I), but (B) is $y^2+exy=A+[aA/(a+A)](e-1)$......$(L)$; hence $x^2y^2=Aa+[a^2A(e-1)/(a+A)]-(ae+A)xy+ex^2y^2$ ... is (IIIa) *by generalizing*, and therefore $x^2y^2-[a+(A+1)/(e-1)]xy+a^2A/(a+A)+aA/(e-1)=0$ is the general form .....(J). $\therefore$ take (A) $x^2=a-xy$ and (B) $y^2=A+d-exy$; and let $e=2$. $\therefore d=(e-1)xy=xy=6$; $A=15$; $y^2+2xy=21$......$(B_{|})$; $(2-1)x^2y^2-41xy+210=0$......(C). $\therefore (xy-(20\frac{1}{2})^2=210\frac{1}{4}$......$(C_{|})$. $\therefore xy=20\frac{1}{2}\mp 14\frac{1}{2}$, *i. e.* $xy=6$......(IV), same as by (I) (II).

But also $x_{||}y_{||}=20\frac{1}{2}+14\frac{1}{2}=35$......$(D)$. $\therefore$ by (A) or (I) $x_{||}{}^2=10-35=-25$; *while* by $(B_{|})$ $y^2=21-70=-49$. $y_{||}=\mp_{|}7\surd(-1)$; $x=\pm_{|}5\surd(-1)$.... (E) roots that *do* satisfy (I) or (A) ; $(B_{|})$ *as well as* $x=\pm_{|}2$; $y=\pm_{|}3$......(V), —and a general sub-ultimate fourth degree (A) $(B_{|})$ has, its eight roots in pairs "where the positive are the negative roots *reversed in signs.*"

Half are roots *both of* (A) $(B_{|})$ and of (I) (II) because (I) is *identically* equation (A). But changing $(B_{|})$ in (II) changes its *second sets* of roots,—to say it *destroys* them is absurd! So therefore (I) (II) have *their own half set* of co-roots besides the roots that (II) shares also with $(B_{|})$ because (I) is *identically* (A). Q. E. D.

As the second set *cannot be unreal*, they *are real* and derive of x^2y^2 *vanished* by *symmetry* which is *why* (IIIa) *cannot* diverge its roots (VI) from (V) as

equation (C) *does* diverge its roots (E) from the roots (V). Therefore, *no other result being possible*, the roots that (II) has with (I) but not with ($B_{|}$) are $x_{|}=\mp_{|}2$; $y_{|}=\mp_{|}3$ and these with $x=\pm_{|}2$; $y=\pm_{|}3$ are *the eight*, and are *all* the roots of (I) (II) a fourth degree *singular* case. Q. E. D.

NOTE.—The $\pm_{|}\ldots\ldots\pm_{|}$; $\pm_{|}\ldots\ldots\mp_{|}$; or $(_{|})\ldots\ldots(_{|})$; mean "change signs in *unison only*." Thus in "SECOND" $_{|}a\ldots\ldots{}_{|}b$ (*not a* prime, etc.)

GEOMETRY.

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81. Proposed by CHAS. C. CROSS, Laytonsville, Md.

A circle is drawn bisecting the lines joining the points of contact of the escribed circles with the sides produced. Another circle is drawn passing through the centers of the circles drawn tangent externally to the in-circle and internally to the sides of the triangle. Prove that the centers of these two circles, the in-center and the circumcircle are collinear.

I. Solution by G. B. M. ZERR. A. M.. Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let ABC be the triangle; O_a, O_b, and O_c the centers of the escribed circles tangent externally to the sides a, b, and c respectively, h and g the points of tangency of circle whose center is O_a with the sides c and b produced; e and f the points of tangency of the circle, center O_b, with the sides c and a produced; d and k the points of tangency of the circle, center O_c, with the sides b and a; O the center of the in-circle; a, b, and c the centers of the circles described tangent to circle, center O, and the sides b and c, c and a, and a and b; E, G, and K the feet of the perpendiculars from the centers a, b, and c to the side a; P, Q, and R the middle points of the lines de, gh, and hk respectively; F, L, and D the feet of the perpendiculars let fall from P, Q, and R on the side a; and O', M', M the centers of the circles through a, b, and c, A, B, and C, and P, Q, and R respectively.

Lemma—The coördinates of the center of a circle passing through (x_1, y_1), (x_2, y_2), (x_3, y_3), are given by

$$\alpha=\frac{(x_1^2+y_1^2)(y_3-y_2)+(x_2^2+y_2^2)(y_1-y_3)+(x_3^2+y_3^2)(y_2-y_1)}{2x_1(y_3-y_2)+2x_2(y_1-y_3)+2x_3(y_2-y_1)}\ldots\ldots\ldots\ldots(1).$$

$$\beta=\frac{(x_1^2+y_1^2)(x_3-x_2)+(x_2^2+y_2^2)(x_1-x_3)+(x_3^2+y_3^2)(x_2-x_1)}{2y_1(x_3-x_2)+2y_2(x_1-x_3)+2y_3(x_2-x_1)}\ldots\ldots\ldots\ldots(2).$$

$Bk=Cf=s-a$, $Cg=Ad=s-b$, $Bh=Ae=s-c$.

Taking B as origin and axes rectangular we get coördinates of M, $\{\frac{1}{2}a, \frac{1}{2}a\cot A\}$; of O, $\{s-b, r\}$; of k, $\{-[s-a], 0\}$; of h, $\{-[s-c]\cos B, -[s-c]\sin B\}$; of f, $\{s, 0\}$; of g, $\{a+[s-b]\cos C, -[s-b]\sin C\}$; of d, $\{s\cos C-a, s\sin C\}$; of e, $\{s\cos B, s\sin B\}$.

$\therefore$ of R, $\{-[\frac{1}{2}(s-a)+\frac{1}{2}(s-c)\cos B],\ -\frac{1}{2}(s-c)\sin B\}$;

of Q, $\{\frac{1}{2}[s+a+(s-b)\cos C],\ -\frac{1}{2}(s-b)\sin C\}$;

of P, $\{\frac{1}{2}[s(\cos C+\cos B)-a],\ \frac{1}{2}s(\sin C+\sin B\}$.

$BH=r\cot\frac{1}{2}B$, $OH=r$, $bE=x$, $BE=x\cot\frac{1}{2}B$.

$\therefore$ $(r-x)^2\cot^2\frac{1}{2}B+(r-x)^2=(r+x)^2$. $\therefore$ $x=r(1-\sin\frac{1}{2}B)/(1+\sin\frac{1}{2}B)$.

Let $r(1-\sin\frac{1}{2}B)/(1+\sin\frac{1}{2}B)=m$, $r(1-\sin\frac{1}{2}C)/(1+\sin\frac{1}{2}C)=n$, $r(1-\sin\frac{1}{2}A)/(1+\sin\frac{1}{2}A)=l$.

$\therefore$ coördinates of b are $(m\cot\frac{1}{2}B,\ m)$; of c, $(a-n\cot\frac{1}{2}C,\ n)$; of a, $(p\cos aBC$, $p\sin aBC)$, where $p=\sqrt{[c^2-2cl\cot\frac{1}{2}B+l^2\mathrm{cosec}^2\frac{1}{2}A]}$, and $\tan aAB=(c\tan B-l\tan B\cot\frac{1}{2}A-l)/(c-l\cot\frac{1}{2}A+l\tan B)$.

Substituting in (1) and (2) the truth of the proposition appears.

By substituting the coördinates of P, Q, R, and a, b, c in (1), (2) we get, after a prodigious amount of work, the coördinates of two points. If the line through these two points coincides with the line through O, M, the proposition is true.

[NOTE.—Dr. Zerr furnished a very beautiful figure to go with his solution, but we lacked the time to engrave it. EDITOR.]

83. Proposed by **WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, O.**

θ being variable, find the locus of a point whose coördinates are

$$a\tan(\theta+\alpha),\quad b\tan(\theta+\beta).$$

Solution by the PROPOSER.

The rectilinear coördinates being x and y, $x=a\tan(\theta+\alpha)$.... .....(1), $y=b\tan(\theta+\beta)$............(2). (1) gives $\theta+\alpha=\tan^{-1}(x/a)$.............(3), $\theta+\beta=\tan^{-1}(y/b)$....(4). Eliminating θ, $\tan^{-1}(x/a)-\tan^{-1}(y/b)=\alpha-\beta$....(5).

Taking tangents of both members of (5) and reducing,

$$xy-\cot(\alpha-\beta)(bx-ay)+ab=0\ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

the equation to the required locus.

Solved in a similar manner by *COOPER D. SCHMITT*, *T. W. PALMER*, *OTTO CLAYTON*, and *G. B. M. ZERR*.

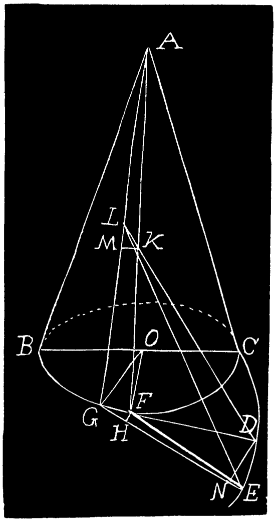
CALCULUS.

65. Proposed by **GEORGE LILLEY, Ph. D., LL. D., Professor of Mathematics, State University, Eugene, Ore.**

A string is wound spirally 100 times around a cone 100 feet high and 2 feet in diameter at the base. Through what distance will a duck swim in unwinding the string keeping it taut at all times, the cone standing on its base and at right angles to the surface of the water ?

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let h=height, r=radius of base, $l=\sqrt{(h^2+r^2)}$=slant height, n=number of spirals ; D, E, two consecutive points in the duck's path ; K, L, corresponding consecutive points of tangency of string to cone ; F, G, corresponding consecutive points in base of cone.



Let $CD=s$, $FK=x$. $\therefore$ $DE=ds$, $ML=dx$.

$ds^2=DN^2+NE^2$ ....(1).

From the similar triangles GOF and DFE,

$r : FG=FD : DN$. $\therefore$ $DN=(FD.FG)/r$..........(2).

From the triangles LMK and KFD,

$dx : MK=x : DF$. $\therefore$ $FD=(x.MK)/dx$...... ....(3).

$MK : (l-x)=GF : l$. $\therefore$ $MK=[(l-x).GF]/l$......(4).

But $CF : x=2\pi rn : l$.

$\therefore$ $GF : dx=2\pi rn : l$. $\therefore$ $GF=(2\pi rn.dx)/l$........(5).

(5) in (4) gives, $MK=[2\pi rn(l-x)dx]/l^2$.....(6).

(6) in (3) gives, $FD=[2\pi rnx(l-x)]/l^2$ ......(7).

(7) and (5) in (2) gives $PN=[4\pi^2r^2n^2(l-x)xdx]/rl^3$..(8).

The increment of FD is $(GH+NE)=FG+NE)$.

$\therefore$ from (7) $d(FD)=[2\pi rn(l-2x)dx]/l^2=(FG+NE)$.(9).

(5) in (9) gives, $NE=[-4\pi rnxdx]/l^2$.......(10).

(8) and (10) in (1) gives,

$$ds=\frac{4\pi rnx}{l^2}\sqrt{1+\frac{\pi^2n^2}{l^2}(l-x)^2}\,dx. \quad \therefore s=\frac{4\pi rn}{l^2}\int_0^l\sqrt{1+\frac{\pi^2n^2}{l^2}(l-x)^2}\,xdx,$$

$$=(4r/3\pi n)+(2r/3)[\pi n-(2/\pi n)]\sqrt{1+\pi^2n^2}+2r\log(\pi n+\sqrt{1+\pi^2n^2}).$$

In the problem $r=1$, $n=100$.

$$\therefore s=(1/75\pi)+\tfrac{2}{3}[100\pi-(1/50\pi)]\sqrt{1+10000\pi^2}+2\log(100\pi+\sqrt{1+10000\pi^2}).$$

$\therefore$ $s=68948.771$ feet.

66. Proposed by J. K. ELLWOOD, A. M., Principal of the Colfax School, Pittsburg, Pa.

Around the top of a conical frustum,—base 5 feet, top 1 foot, altitude 100 feet,—is wound a rope 100 feet long and 1 inch thick. It is unwound by a hawk flying in one plane. How far does Mr. Hawk fly ?

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Instead of measuring the thickness of the rope on the slant height, in this solution it is measured on the altitude, the difference being for the whole winding not above .007 of an inch. Measuring from middle of rope to middle of rope, the first coil is $\frac{1}{2}$ inch below top of frustum, second coil $\frac{3}{2}$ inches, third $\frac{5}{2}$ inches, etc. Let x=height of cone.

$\therefore$ $x : \frac{5}{2}=x-100 : \frac{1}{2}$, $\therefore$ $x=125$.

$125-100=25$ feet, $=300$ inches.

$\therefore 300 : 6 = 300\frac{1}{2} : 6\frac{1}{100}$,

$300 : 6 = 301\frac{1}{2} : 6\frac{3}{100}$,

$300 : 6 = 302\frac{1}{2} : 6\frac{5}{100}$.

. . . .

Let n=number of turns.

$\therefore 2\pi(6\frac{1}{100}+6\frac{3}{100}+6\frac{5}{100}+\text{to } n \text{ terms})=1200$ inches.

$\therefore 2\pi(6+6+6+\ldots\ldots)+2\pi(\frac{1}{100}+\frac{3}{100}+\frac{5}{100}+\ldots\ldots)=1200$.

$\therefore 12\pi n+\pi n^2/1200=100$. $\therefore n=31.03$ coils. From the previous problem,

$$s=\frac{4\pi rm}{l^2}\int_{25}^{27\frac{7}{12}}\sqrt{1+\frac{\pi^2 m^2}{l^2}(l-x)^2}.xdx, \text{ where } m=331,$$

the number of coils of rope to top of cone beginning with the bottom coil, and $l=331$ inches the height$=27\frac{7}{12}$ feet; also let 25 feet$=l'$, $r=\frac{331}{600}$ feet=radius of base of cone at bottom of rope.

$$\therefore s=\frac{4\pi rm}{l^2}\int_{l'}^{l}\sqrt{1+\frac{\pi^2 m^2}{l^2}(l-x)^2}.xdx.$$

$$\therefore s=\frac{4r}{3\pi m}+\frac{2r}{3l^3\pi m}[(l^2+ll'-2l'^2)\pi^2 m^2-2l^2]\sqrt{l^2+(l-l')^2\pi^2 m^2}$$

$$+2r\log\left(\frac{(l-l')\pi m+\sqrt{l^2+(l-l')^2\pi^2 m^2}}{l}\right).$$

$$s=(1/450\pi)+(1/900\pi)(28861\pi^2-2)\sqrt{1+961\pi^2}+\tfrac{331}{300}\log(31\pi+\sqrt{1+961\pi^2}).$$

$s=9817.69235$ feet.

In this solution the rope is unwound from the bottom.

MECHANICS.

55. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Miss.

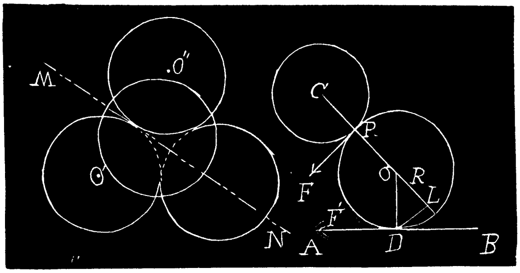
Three equal heavy spheres, each of weight W, are placed on a rough ground just not touching each other. A fourth sphere of weight nW is placed on the top touching all three. Show that there is equilibrium if the coefficient of friction between two spheres is greater than $\tan\frac{1}{2}\alpha$, and that between a sphere and the ground is greater than $n\tan\frac{1}{2}\alpha/(n+3)$, where α is the inclination to the vertical of the straight line joining the centers of the upper and one lower sphere.

Solution by the PROPOSER.

I.

In what different ways may motion occur, if the friction be such as to permit motion?

So long as the lower spheres maintain their positions, the upper will not move.



PLAN. SECTION MN.

How, then, may the lower move? Referring to the section—

1. By sliding in the direction BA;
2. By rolling in the direction BA;
3. By rolling in the direction AB;
4. By sliding in the direction AB.

Motions (1) and (2) cannot take place because, even if this did not at once bring the lower spheres into contact, such would necessarily raise the upper sphere and, evidently, there are no forces acting which could have this effect. Having disposed of (1) and (2), consider (3).

The forces acting on the sphere whose center is O are its own weight, the friction F', the friction F, and the normal action between the two spheres, R. Rolling in the direction AB will not occur unless the moment of R about D is greater than that of F. Equating these two moments,

$$F.PL=R.DL,$$

$$F=(DL/PL)R$$

$$=\tan\tfrac{1}{2}\alpha.R,$$

since $\angle DPL=\frac{1}{2}\angle DOL$, and $\angle DOL=\alpha$, OD being vertical and OL the prolongation of CO.

If μ=coefficient of friction between the two spheres, $F=\mu R$.

$\therefore\ \mu=\tan\frac{1}{2}\alpha$.

If, then, the coefficient of friction between the spheres whose centers are C and 0 is greater than tan $\frac{1}{2}\alpha$, F will prevent motion (3).

Now consider (4).

This will take place unless F' is equal to, or greater than, $R\sin\alpha-F\cos\alpha$.

The normal pressure on the ground at D is the weight of sphere O plus one-third of the weight of sphere C, or $W+\frac{1}{3}nW$.

If μ' is the coefficient of friction between the sphere and the ground, $F'=\mu'(n+3)\frac{1}{3}W$, and, we have, for equilibrium,

$$\mu'(n+3)\tfrac{1}{3}W=R\sin\alpha-F\cos\alpha$$

$$=R(\sin\alpha-\tan\tfrac{1}{2}\alpha\cos\alpha),$$

taking the least value of F consistent with equilibrium. From this,

$$R=\frac{\mu'(n+3)}{\tan\frac{1}{2}\alpha}\cdot\tfrac{1}{3}W.$$

Consider the forces acting on the upper sphere—its own weight nW, F, and R reversed in direction, and, in addition, the actions F and R from each of the two other spheres, O' and O'' (see plan). Resolving vertically,

$$nW=3R\cos\alpha+3F\sin\alpha$$
$$=3R(\cos\alpha+\tan\tfrac{1}{2}\alpha\sin\alpha)=3R\,;$$
$$R=\tfrac{1}{3}nW.$$

Equating the two values of R and solving for μ',

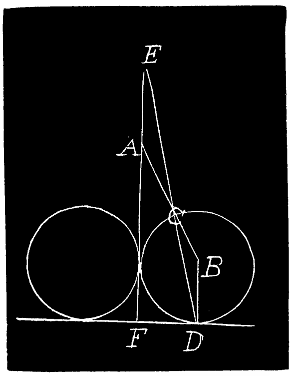
$$\mu'=\tan\tfrac{1}{2}\alpha\,.\frac{n}{n+3}.$$

REMARKS. Looking at the section it might seem, at a glance, that R ought to bs $\frac{1}{3}nW\cos\alpha$, whereas it has been found to be $\frac{1}{3}nW$. It must be remembered that the two spheres, C and O, are not in equilibrium by themselves, C being held in position by reactions and friction from O' and O'' as well as from O.

It may be noted, too, that the friction brought into play at P may not be sufficient to prevent sliding. In order that motion of this kind may not take place the coefficient of friction must not be less than $\tan\alpha$. But the reason why C does not slide is that it cannot descend unless O rolls or slides out of its way toward B. And, as has been seen, neither of these motions can occur if $\mu>\tan\frac{1}{2}\alpha$ and $\mu'>\tan\frac{1}{2}\alpha\,.\,n/(n+3)$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let W=weight of one of the equal spheres, W_1=weight of upper sphere, $\angle FAB=\alpha$.



Since sphere B and reaction of the ground acts through D, the total resistance between spheres A and B must act through D.

Hence the total resistance between A and B acts in the line ECD.

$\therefore$ The angle of friction between A and B must be $\geqq$ than $\angle ACE=\frac{1}{2}\alpha$, $\therefore$ coefficient of friction $\geqq\tan\frac{1}{2}\alpha$.

Let μ=coefficient of friction between B and the ground, R=total normal, P=total tangential resistanee at D.

$\therefore\ \mu R=P$ or $\mu=P/R$.

$\therefore\ \mu=(\frac{1}{3}W_1\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha)/[(\frac{1}{3}W_1+W)\cos^2\frac{1}{2}\alpha]$.

$$\therefore\ \mu=\frac{W_1}{W_1+3W}\tan\tfrac{1}{2}\alpha=\frac{n}{n+3}\tan\tfrac{1}{2}\alpha,\ \text{when } W_1=nW.$$

Let $AC=b$, $CB=c$. Then $\sin\alpha=\dfrac{2c}{\sqrt{3}(b+c)}$.

$$\therefore\ \tan\tfrac{1}{2}\alpha=\frac{(b+c)\sqrt{3}-\sqrt{3b^2+6bc-c^2}}{2c}$$

$\therefore\ \tan\frac{1}{2}\alpha=\sqrt{3}-\sqrt{2}$ when $b=c$.

NOTE ON SECOND SOLUTION OF PROBLEM 49, MECHANICS.

By ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Miss.

My work as published in the June-July number is the solution of the following problem :

A uniform bar, whose other dimensions may be neglected in comparison with its length ($2a$), is placed in vertical position on the edge of a platform of given height and, being slightly displaced from the vertical, is caused to revolve about the edge as an axis, the lower end being set free when the bar becomes horizontal. Where and in what manner will it strike the ground ?

The discussion of either problem—the one just stated or the one actually proposed—naturally divides itself into two parts, the first dealing with the motion before, the second after, the stick leaves the platform. It was the latter that chiefly occupied my thoughts when I prepared the solution referred to above.

I now offer the following as a discussion of the first part :

Let m=mass of stick,

$2b$=length of stick,

$2a$=other dimension perpendicular to edge of platform,

θ=angle through which stick has turned in t seconds,

ω=angular velocity at time t.

$$\frac{d^2\theta}{dt^2}=\frac{mgb\sin\theta}{m\{[(a^2+b^2)/3]+b^2\}}=\frac{3b\sin\theta}{a^2+4b^2}g\,;\ \omega^2=\left(\frac{d\theta}{dt}\right)^2=-\frac{6b\cos\theta}{a^2+4b^2}g+c,$$

or, since when $\theta=0$, $\omega=0$,

$$\omega^2=\frac{6bg}{a^2+4b^2}(1-\cos\theta.)\quad \text{Normal acceleration}=b\omega^2=\frac{6b^2g}{a^2+4b^2}(1-\cos\theta).$$

Component of acceleration due to gravity directly opposite to this$=g\cos\theta$.

At the instant these become equal the contact between stick and platform ceases.

This occurs when $\cos\theta=(6b^2)/(a^2+10b^2)$.

Up to this time (friction being supposed sufficiently great to prevent sliding) the stick has turned about the edge of the platform. After leaving the platform the center of gravity of the stick describes a parabola, and the stick revolves with a constant angular velocity, ω_1 about its center of gravity. This part of the motion might be discussed as in the June-July number.

DIOPHANTINE ANALYSIS.

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58. Proposed by E. S. LOOMIS, Ph. D., Professor of Mathematics in Cleveland West High School; Berea, O.

The base of a right-angled triangle is 105; find all the perpendiculars and hypotenuses to fit it, such that their values shall be integers.

I. Solution by the PROPOSER.

1. Let the hypotenuses and sides of a right-angled triangle be represented by m^2+n^2, m^2-n^2, and $2mn$.

2. Since in this problem we have to do with squares, we will draw a few observations from the squares of the integers, 1 to 18.

Int.:	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15,	16,	17,	18.
Sq.:	1,	4,	9,	16,	25,	36,	49,	64,	81,	100,	121,	144,	169,	196,	225,	256,	289,	324.

3. Observations: (1) The difference between two consecutive squares is odd, the smallest difference being 3; therefore 105 is the difference between two consecutive squares; (2) The series of differences increases by 2; (3) Any difference is two times the number of the place of the lesser square in the series, $+1$, therefore let x=the number of the place of the lesser square; whence $2x+1=105$, or $x=52$, therefore all pairs of integers sought are confined to the series of squares of the integers 1 to 53; (4) The difference between squares in odd, or even, places is even; as 105, and all its factors, are odd, one must seek for differences of two squares, one occupying an odd, and the other occupying an even place.

4. Since $105=1\times105=3\times35=5\times21=7\times15=15\times7=21\times5=35\times3=105\times1$, there are eight cases to consider, each case giving 0, 1, or more than 1 pair of integers.

5. Since 105 is odd, we must have $m^2-n^2=105$.

6. Solution of cases: (1) Let $m^2-n^2=1\times105$, therefore $m=\sqrt{(105+n^2)}$. Since n is less than m, values for n, by observation (3), will be integers less than 53. Therefore by inspection, when

$$\begin{aligned} n&=4, \quad 8, \quad 16, \quad 52, \\ m&=11, \quad 13, \quad 19, \quad 53, \\ m^2+n^2&=137, \quad 233, \quad 617, \quad 5313, \\ \text{and } 2mn&=88, \quad 208, \quad 608, \quad 5512. \end{aligned}$$

Therefore there are four pairs of integers, for $1\times105=(11^2-4^2)=(13^2-8^2)=(19^2-16^2)=(53^2-52^2)$.

(2) Let $3(m^2-n^2)=3\times35$; whence $m=\sqrt{(35+n^2)}$; by observation (3) $n<17$, and by inspection, $105=3\times35=3(6^2-1^1)=3(18^2-17^2)$. When $n=1$, 17; $m=6$, 18; $3(m^2+n^2)=111$, 1839; and $3(2mn)=36$, 1836, giving two pairs for 3×35.

(3) Let $105=5(m^2-n^2)=5\times21=5(5^2-2^2)=5(11^2-10^2)$; whence $m=\sqrt{(21+n^2)}$. When $n=2$, 10; $m=5$, 11; $5(m^2+n^2)=145$, 1105; and $5(2mn)=100$, 1100, giving two pairs.

(4) Let $105=7\times15=7(m^2-n^2)=7(4^2-1^2)=7(8^2-7^2)$; whence $m=\sqrt{(15+n^2)}$. Therefore $7(m^2+n^2)=119$, 791; and $7(2mn)=56$, 784, giving two pairs.

(5) Let $105=15\times7=15(m^2-n^2)=15(4^2-3^2)$; whence $m=\sqrt{(7+n^2)}$, and $15(m^2+n^2)=375$, and $15(2mn)=360$, giving one pair.

(6) Let $105=21\times5=21(m^2-n^2)=21(3^2-2^2)$; whence $m=\sqrt{(5+n^2)}$, and $21(m^2+n^2)=273$, and $21(2mn)=252$, giving one pair.

(7) Let $105=35\times3=35(m^2-n^2)=35(2^2-1^2)$; whence $m=\sqrt{(3+n^2)}$, and $35(m^2+n^2)=175$, and $35(2mn)=140$, giving one pair.

(8) Let $105=105\times1=105(m^2-n^2)$. But 1 can not be the difference between two squares, by observation (1). Hence for $105=105\times1$ there is no solu-

tion. Therefore there can be but 13 pairs of integers which will satisfy the conditions of the problem, and the hypotenuses are 137, 233, 617, 5513, 111, 1839, 145, 1105, 119, 791, 375, 273, and 175; and the perpendiculars are 88, 208, 5512, 36, 1836, 100, 1100, 56, 784, 360, 252, and 140.

The principles established and employed in the above problem will hold whenever the given number is odd. But if the number is even, say 104, then from the series of squares above new observations must be drawn and employed.

II. Solution by SYLVESTER ROBBINS, North Branch, New Jersey.

Every rational right-angled triangle having 105 for its base must have an even number for its perpendicular and an odd one for its hypotenuse. Notice that of the factors entering into 105, each of these, 3, 5, 7, is difference of two squares *once*; each of the factors, 15, 21, 35, is the difference of two squares *twice*; and 105 itself is the difference of two squares *four times*. The perpendicular in each of these triangles must be twice the *product* of two roots, the base must be the difference of their squares, and the hypotenuse must be the *sum* of the same two squares. In order to obtain the *greatest* number of rational right triangles having same base, the n factors of said base must be prime numbers, never using 2, then $(3^n-1)/2$ will express the number of triangles, 1, 4, 13, 40, 121, 364, 1093, 3280, etc. In this problem there are $3\times1+3\times2+1\times4=13$ triangles. The following are general expressions for sides of rational right triangles. Roots: $r=2\sqrt{35}$, $3\sqrt{21}$, $4\sqrt{15}$, $6\sqrt{3}$, $18\sqrt{3}$, $5\sqrt{5}$, $11\sqrt{5}$, $4\sqrt{7}$, $8\sqrt{7}$, 11, 13, 19, 53; $s=1\sqrt{35}$, $2\sqrt{21}$, $3\sqrt{15}$, $1\sqrt{3}$, $17\sqrt{3}$, $2\sqrt{5}$, $10\sqrt{5}$, $1\sqrt{7}$, $7\sqrt{7}$, 4, 8, 16, 52; base $=r^2-s^2=105$ in each case; perpendiculars$=2rs=140$, 252, 360, 36, 1836, 100, 1100, 56, 784, 88, 208, 608, 5512; $h=r^2+s^2=175$, 273, 375, 111, 1839, 145, 1105, 119, 791, 137, 233, 617, 5513, being the 13 values in order, respectively.

III. Solution by JOSIAH H. DRUMMOND; LL. D., Portland, Me., and G. B. M. ZERR, A. M., Ph. D., The Russell College. Lebanon, Va.

Let h=hypotenuse, p=perpendicular. Then $h^2-p^2=(105)^2$ or $h+p=(105)^2/(h-p)$. $\therefore$ $h-p$ must be some factor of $(105)^2$; $h+p$ is also; and as the factors are all odd, we have integral values of h and p for every integral factor, and, moreover, for all factors less than 105, these values will be positive. The prime factors are 1, 3, 3, 5, 5, 7, 7; and the factors less than 105, of $h-p$, are readily found to be 1, 3, 5, 7, 9, 13, 21, 25, 35, 45, 49, 63, 75; the corresponding factors, or values of $h+p$ are 11025, 3675, 2205, 1575, 1225, 735, 525, 441, 315, 245, 225, 175, 147. Hence, $h=5513$, 1839, 1105, 791, 617, 375, 273, 233, 175, 145, 137, 119, 111; and $p=5512$, 1836, 1100, 784, 608, 360, 252, 208, 140, 100, 88, 56, 36; and $b=105$ in each case.

M. A. Gruber, A. M., War Department, Washington, D. C., finds the 13 sets of values as given above, and refers to the full explanation of finding these results as given in his paper on "Integral Sides of Right Triangles," pages 106—108, of Vol. IV, No. 4 (April, 1897), of Monthly, where the problem, "Given one of the legs of a right-triangle of integral sides to find the other leg and the hypotenuse," is fully treated.

A. H. BELL, Hillsboro, Ill., finds the 13 sets of values, and deduces the general rule for finding the same from the principle shown in his solution of Problem 31, page 363, Vol. II, of MONTHLY. P. S. BERG, A. M., Principal of Schools, Larimore, N. D., sends a good solution finding the 13 sets of values by the method of solution No. I, of the problem last referred to.

WILLIAM HOOVER, A. M., Ph. D., Ohio University, Athens, Ohio, solves substantially as in Solution I, above, but as he did not carry out the work far enough to give all the results, we adopted the phraseology of the Proposer.

Also excellently solved, but without determining the full sets of results, by OTTO CLAYTON, A. B., Remington, Ind.; J. O. MAHONEY, B. E., M. Sc,. Lynnville, Tenn.; and O. W. ANTHONY, M. Sc., Instructor in Mathematics in Boys' High School, New York City.

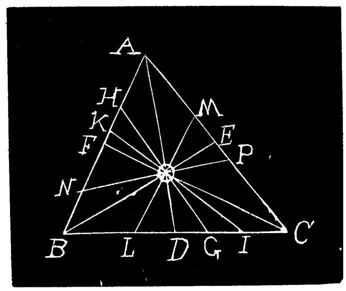
AVERAGE AND PROBABILITY.

56. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find the chance that the center of gravity of a triangle lies inside the triangle formed by three points taken at random within the triangle. [From *Williamson's Integral Calculus.*]

Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

Let D, E, and F be the middle points of the respective sides of the triangle. Then O, the common intersection of HD, BE, and CF, is the center of gravity of the triangle. Suppose that the first point is somewhere upon the line OI, and the second somewhere upon the line OG. Put $DI=x$ and $DG=y$. Then the favorable positions for the third point are upon the surface OKH, whose area is



$$\frac{\triangle}{3}\left(\frac{4x}{a+6x}-\frac{4y}{a+9y}\right)$$

y being less than x, and x less than $\frac{1}{2}a$.

If $LD=y$, and the second point be upon OL, the favorable positions for the third point are upon the surface $OKAMO$, whose area is

$$\frac{\triangle}{3}\left(\frac{4x}{a+6x}+\frac{4y}{a+6y}\right).$$

If $FN=y$ and the second point be upon the line ON, the favorable positions for the third point are upon the surface $OKAPO$, whose area is

$$\frac{\triangle}{3}\left(\frac{4x}{a+6x}+1-\frac{4y}{a+6y}\right).$$

If G be taken between I and C, H will fall between F and K, and the favorable positions for the third point will be upon OHK, whose area is

$$\frac{\triangle}{3}\left(\frac{4y}{a+6y}-\frac{4x}{a+6x}\right).$$

The chance that the first point falls upon the element of surface at OI is $(\triangle/3)(dx/a)/\triangle = dx/3a$.

The chance that the second falls upon the element of surface at OG or OL is $dy/3a$. The chance that it falls upon the element at ON is $dy/3c$.

The chance that the third point falls upon any surface S is $S/\triangle$.

Hence if the first point be confined to the surface DOC and the second to surface CFB, the required chance is

$$P_1 = \frac{1}{27a^2}\Bigg[\int_0^{\frac{1}{2}a}\int_0^{x}\left(\frac{4x}{a+6x}-\frac{4y}{a+6y}\right)dxdy + \int_0^{\frac{1}{2}a}\int_0^{y}\left(\frac{4y}{a+6y}-\frac{4x}{a+6x}\right)dydx$$

$$+\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}a}\left(\frac{4x}{a+6x}+\frac{4y}{a+6y}\right)dxdy\Bigg] + \frac{1}{27ac}\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}c}\left(\frac{4x}{a+6x}+1-\frac{4y}{c+6y}\right)dxdy.$$

By combining symmetrical expressions this reduces to

$$P_1 = \frac{8}{27a^2}\Bigg[\int_0^{\frac{1}{2}a}\int_0^{x}\left(\frac{x}{a+6x}\right)dxdy - {}^{*}\int_0^{\frac{1}{2}a}\int_0^{y}\left(\frac{x}{a+6x}\right)dydx$$

$$+\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}a}\left(\frac{x}{a+6x}\right)dxdy\Bigg] + \frac{4}{27ac}\Bigg[\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}c}\left(\frac{x}{a+6x}\right)dxdy$$

$$-\int_0^{\frac{1}{2}c}\int_0^{\frac{1}{2}a}\left(\frac{y}{c+6y}\right)dydx + \tfrac{1}{4}\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}c}dxdy\Bigg]$$

$$=\frac{8}{27a^2}\left[\int_0^{\frac{1}{2}a}\left(\frac{x^2}{a+6x}\right) - \frac{x(\frac{1}{2}a-x)}{a+6x} + \tfrac{1}{2}a\left(\frac{x}{a+6x}\right)\right]$$

$$+\frac{4}{27ac}\left[\int_0^{\frac{1}{2}a}\tfrac{1}{2}c\left(\frac{x}{a+6x}\right)dx - \int_0^{\frac{1}{2}c}\tfrac{1}{2}a\left(\frac{y}{c+6y}\right)dy + \tfrac{1}{2}c\int_0^{\frac{1}{2}a}dx\right]$$

$=\frac{13}{972}+\frac{4}{729}\log 2$. Since this is independent of a, b, and c, it is evident that if the second point had been allowed to occupy all positions on both sides of CF, the result would have been twice as great, and if the first point had been allowed to take all positions instead of being confined to DOC the result would have been six times as great.

Hence the required probability is $P=12P_1=\frac{13}{81}+\frac{16}{243}\log 2=\frac{1}{81}[13+\frac{16}{3}\log 2]$ $=.20613$.

Remark. In integrating the expression marked *, I reversed the order of integration. Thus,

$$\int_0^{\frac{1}{2}a}\int_0^{y}\left(\frac{x}{a+6x}\right)dydx = \int_0^{\frac{1}{2}a}\int_x^{\frac{1}{2}a}\left(\frac{x}{a+6x}\right)dxdy = \int_0^{\frac{1}{2}a}\left(\frac{x(\frac{1}{2}a-x)}{a+6x}\right)da.$$

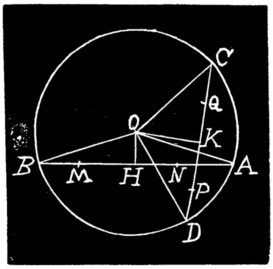
[The result given in *Williamson's Calculus* is $\frac{1}{27}(2+\frac{10}{3}\log 4$. Editor.]

57. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.**

A chord is drawn through two points taken at random in the surface of a circle. If a second chord be drawn through two other points taken at random in the surface, find the chance that the quadrilateral formed by joining the extremities of the two chords will contain the center of the circle.

I. **Solution by the PROPOSER.**

Let M, N be the first two random points; AB the chord through them; P, Q the second two random points; C, D the chord through them; and O the center of the circle. Draw OH, OK perpendicular to AB, CD.



Let $AO=r$, $AM=w$, $MN=x$, $CP=y$, $PQ=z$, $\angle AOH=\theta$, $\angle COK=\varphi$, $\angle AOC=\psi$, and μ=the angle AB makes with some fixed line.

Then $AH=r\sin\theta$, $CK=r\sin\varphi$; an element of the circle at M is $r\sin\theta d\theta dw$; at N, $d\mu x dx$; at P, $r\sin\varphi d\varphi dy$; at Q, $d\psi z dz$. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and θ, and doubled; of ψ, $\pi-2\theta$ and π, and doubled; of μ, 0 and 2π; of w, 0 and $2r\sin\theta=s$; of x, 0 and w, and doubled; of y, 0 and $2r\sin\varphi=v$; and of z, 0 and y, and doubled. Hence, since the whole number of ways the four points can be taken is $\pi^4 r^8$, the required chance is

$$p=\frac{16}{\pi^4 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\int_0^{w}\int_0^{v}\int_0^{y} r\sin\theta d\theta r\sin\varphi d\varphi d\psi d\mu dw x dx dy z dz$$

$$=\frac{8}{\pi^4 r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\int_0^{w}\int_0^{v}\sin\theta\sin\varphi d\theta d\varphi d\psi d\mu dw x dx y^2 dy$$

$$=\frac{64}{3\pi^4 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\int_0^{w}\sin\theta\sin^4\varphi d\theta d\varphi d\psi d\mu dw x dx$$

$$=\frac{32}{3\pi^4 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\sin\theta\sin^4\varphi d\theta d\varphi d\psi d\mu w^2 dw$$

$$=\frac{256}{9\pi^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\sin^4\theta\sin^4\varphi d\theta d\varphi d\psi d\mu$$

$$=\frac{512}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\sin^4\theta\sin^4\varphi d\theta d\varphi d\psi=\frac{1024}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\theta\sin^4\theta\sin^4\varphi d\theta d\varphi$$

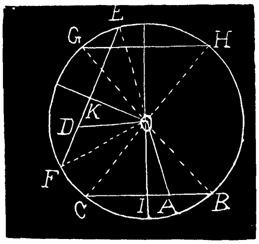
$$=\frac{128}{9\pi^3}\int_0^{\frac{1}{2}\pi}(3\theta^2-3\theta\sin\theta\cos\theta-2\theta\sin^3\theta\cos\theta)\sin^4\theta d\theta=\frac{2}{3}+\frac{139}{72\pi^2}.$$

II. **Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.**

Let A and D represent the positions of the first and third points, and sup-

pose the second and fourth to be taken somewhere upon the lines BC and EF, respectively.

Put $OC=a$, $OA=x$, $OD=y$, $\angle OAC=\theta$, $\angle IOC=\phi$, $\angle ODK=\psi$, and $\angle KOF=\omega$.



Then $a\cos\phi=x\sin\theta$, and $a\cos\omega=y\sin\psi$.

If either E or F is upon the arc HG, the quadrilateral formed by joining the extremities of the chords will contain the center of the circle.

Since the arc HG=the arc CB, the probability of this is $(\phi+\omega)/\pi$.

If $x>0$ and $<a$ the chance that the first point is taken between x and $x+dx$ is $2\pi xdx/a^2\pi=2xdx/a^2$.

If $y>0$ and $<a$ the chance that the third point is taken between the distances y and $y+dy$ is $2ydy/a^2$.

If $\theta>0$ and $<\frac{1}{2}\pi$ the chance that the second point is taken between the line BC and a second line making at A the angle $d\theta$ is $\frac{1}{2}(AC^2+AB^2)d\theta/\frac{1}{2}a^2\pi$ $=2[a^2+x^2(1-2\sin^2\theta)]d\theta/a^2\pi=2(1-2\cos^2\phi+\cos^2\phi\operatorname{cosec}^2\theta)d\theta/\pi$.

If $\psi>0$ and $<\frac{1}{2}\pi$ the chance that the fourth point is taken between the line EF and a second line making at D the angle $d\psi$ is $2(1-2\cos^2\omega+\cos^2\omega\operatorname{cosec}^2\psi)d\psi/\pi$.

If we suppose θ constant while x and ϕ vary, $xdx=-a^2\sin\phi\cos\phi\operatorname{cosec}^2\theta d\phi$. When $x=0$, $\phi=\frac{1}{2}\pi$, and when $x=a$, $\phi=\frac{1}{2}\pi-\theta$. Hence the limits of integration for ϕ are $\frac{1}{2}\pi-\theta$ and $\frac{1}{2}\pi$. If we integrate first with respect to θ, the limits for θ are $\frac{1}{2}\pi-\phi$ and $\frac{1}{2}\pi$, while for ϕ they are 0 and $\frac{1}{2}\pi$.

In like manner we may substitute $-a^2\sin\omega\cos\omega\operatorname{cosec}^2\psi d\omega$ for ydy and integrate between limits $\frac{1}{2}\pi-\omega$ and $\frac{1}{2}\pi$ for ψ, and between 0 and $\frac{1}{2}\pi$ for ω.

Hence the required probability is

$$P=\frac{16}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\phi}^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\omega}^{\frac{1}{2}\pi}(\phi+\omega)(1-2\cos^2\phi+\cos^2\phi\operatorname{cosec}^2\theta)\times$$

$$(1-2\cos^2\omega+\cos^2\omega\operatorname{cosec}^2\psi)\sin\phi\cos\phi\operatorname{cosec}^2\theta\sin\omega\cos\omega\operatorname{cosec}^2\psi d\phi d\omega d\theta d\psi$$

$$=\frac{256}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\phi+\omega)\sin^4\phi\sin^4\omega\, d\phi d\omega$$

$$=\frac{4}{9\pi^3}\int_0^{\frac{1}{2}\pi}(12\pi\phi+3\pi^2+16)\sin^4\phi d\phi=\tfrac{1}{2}+(8/3\pi^2)=.77019.$$

MISCELLANEOUS.

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55. Proposed by J. M. COLAW, A. M., Monterey, Va.

Multiply 6 by 4. Is the problem legitimate when both symbols represent pure number?

[NOTE. "A measured or numbered quantity may be divided into a number of parts, or taken a number of times; but no number can be multiplied or divided into parts."—*McLellan and Dewey's Psychology of Number*. "The astounding thesis is maintained that number is not a magnitude, does not possess quantity at all, and that 'no number can be multiplied or divided into parts'."—*Lefevre's Number and Its Algebra*.]

I. Comment by **DAVID EUGENE SMITH, Ph. D.**, Professor of Mathematics, Michigan State Normal School, Ypsilanti, Mich.

This question seems to me to have no interest to mathematicians. It simply means that somebody has set up a narrow definition of multiplication and has then said, "*Your* work is not multiplication because it does not fit *my* definition." I expressed my humble opinion in the MONTHLY some time ago when the antiquated definition of division was brought up to prove that it was impossible to divide $10 by 2. Such narrow limitations seem to me utterly nonsensical.

In a similar sense we cannot multiply by -1, and we cannot have "imaginary numbers," and 1 is not a number, etc., etc. Mathematical progress has always been made the more difficult because somebody has insisted on hanging on to some antiquated definition.

What do these people who say that we cannot multiply 2 by 3 say to some such simple formula as $e^{\pi i}=-1$? I suppose they say that e, π, i, and -1 have no existence.

II. Comment by **G. B. M. ZERR, A. M., Ph. D.**, President and Professor of Mathematics, The Russell College, Lebanon, Va.

There may be a modern idea that since neither 6 nor 4 is a numbered quantity, the operation is impossible. If it were possible to get the evidence of all the mathematicians, I am sure not one could be found who did not learn his multiplication table, in fact, get his aptness in numbers by the same process as given in the problem. There may be some who claim otherwise but I would even doubt their claim.

$6\times4=24$ is good arithmetic. It seems a pity that vandals should make incursions upon the sacred shrines of Newton, La Place, Pierce, and other noted men of numbers, and so desecrate their immortal works, as to try and mistify their teachings. The God of Mathematics will not permit it.

III. Remarks by **J. K. ELLWOOD, A. M.**, Principal of the Colfax School, Pittsburg, Pa.

Do $4\times6=24$? Can 6 be multiplied by 4 ? Six what ? If 6 units of quantity, yes; but if not a magnitude,—well, what then is it ? "Six" apart from the universe of space, time, and matter, suggests to the mind—what ? The "how many"? The *ratio* of the "how many" to the unit ? Six in the abstract —a pure number—can not, in an arithmetical sense, be multiplied by any abstraction. In an algebraic sense, $4\times6=24$, just as $x\times x=x^2$. That is, we operate with *symbols*, neglecting the realities represented. If two abstract numbers can be multiplied one by the other, why not two concrete numbers, as feet$\times$feet =square feet ?

42. Proposed by **E. B. ESCOTT**, Cambridge, Mass.

To find triangles whose sides and median lines are commensurable.

II. Solutions communicated to "L' Intermediaire des Mathematiciens" (January, 1898) by the **PROPOSER.** Selected and translated by **J. M. COLAW, A. M.**, Monterey, Va.

First Solution. By Chas. Gill (New York, 1848).

$x=t[1-(\cos A+\sin A)(\cos B+\sin B)]$, $y=t[\cos B-\sin B+(\cos A-\sin A)(\cos B+\sin B)]$, $z=t[\cos A-\sin A+(\cos B-\sin B)(\cos A+\sin A)$, whence there exists one of the four relations following:

$\tan\frac{1}{2}A=(16+13\sin B-8\cos B-5\sin 2B-2\sin^2 B)/[(2+\sin B+2\cos B)(5+4\sin B-4\cos B)]$......(I) ; $\tan\frac{1}{2}A=[(1-\sin B)(5-\cos B+4\sin B)]/[(\sin B+\cos B)(1+3\cos B-\sin B)]$......(II) ; $\cot\frac{1}{2}A=-(16-13\cos B+8\sin B-5\sin 2B-2\cos^2 B)/[(2-\cos B-2\sin B)(5+4\sin B-4\cos B)$......(III) ; $\cot\frac{1}{2}A=[(1+\cos B)(5+\sin B-4\cos B)]/[(\sin B+\cos B)(1+\cos B-3\sin B)]$......(IV). Letting, in formula (II), $\sin B=\frac{4}{5}$, $\cos B=-\frac{3}{5}$, then $\tan\frac{1}{2}A=-\frac{11}{2}$, $x=262$, $y=316$, $z=254$, $m_x=261$, $m_y=204$, and $m_z=255$.

Second Solution. From *The Gentleman's Mathematical Companion*, London, 1824, page 350.

Let $z=x+y-d$, then $4m_z{}^2=x^2-2xy+y^2+2d(x+y)-d^2$; $4m_y{}^2=4x^2+4xy+y^2-4d(x+y)+2d^2$; $4m_x{}^2=x^2+4xy+4y^2-4d(x+y)+2d^2$.

Let $4x^2+4xy+y^2-4d(x+y)+2d^2=(2x+y-m)^2$; then $x=(2d^2-4dy+2my-m^2)/(4d-4m)$, $y=(2d^2-4dx+4mx-m^2)/(4d-2m)$.

Let $x^2+4xy+4y^2-4d(x+y)+2d^2=(x+2y-n)^2$; then $x=(2d^2-4dy+2ny-n^2)/(4d-2n)$; $y=(2d^2-4dx+2nx-n^2)/(4d-4n)$.

$x=[d^2(4n-2m)+2d(m^2-n^2)-mn(2m-n)]/[4d(m+n)-6mn]$;

$y=[d^2(4m-2n)-2d(m^2-n^2)-mn)2n-m)]/[4d(m+n)-6mn]$;

$z=x+y-d=-[2d^2(m+n)-6mnd+mn(m+n)]/[4d(m+n)-6mn]$.

We may neglect the common denominator $4d(m+n)-6mn$. We then have to satisfy the condition, $2x^2+2y^2-z^2=36d^4(m-n)^2-24d^3(m+n)(2m^2-5mn+2n^2)+4d^2(4m^4+7m^2n-39m^2n^2+7mn^3+4n^4)-12dmn(m+n)(2m^2-5mn+2n^2)+9m^2n^2(m-n)^2=$a square.

We have also $=\{6d^2(m-n)-2d[(m+n)/(m-n)](2m^2-5mn+2n^2)-3mn(m-n)\}^2$.

Whence $d=[3(m+n)(m-n)^2]/(5m^2-8mn+5n^2)$.

Letting $m=3$, $n=2$, $d=\frac{15}{17}$, $x=656$, $y=414$, $z=290$,
$m_x=142$, $m_y=463$, $m_z=529$,

$m=3$, $n=1$, $d=\frac{24}{13}$, $x=174$, $y=170$, $z=136$,
$m_x=127$, $m_y=131$, $m_z=158$,

$m=3$, $n=-1$, $d=\frac{48}{37}$, $x=650$, $y=318$, $z=628$,
$m_x=377$, $m_y=619$, $m_z=404$,

$m=5$, $n=4$, $d=\frac{3}{5}$, $x=892$, $y=554$, $z=954$,
$m_x=640$, $m_y=881$, $m_z=569$.

[*E. Fauquembergue* says, (*L' Intermediaire*, *Mars* 1897), that Euler, at different times busied himself with the problem of finding a triangle whose sides and medians are commensurable. His solution is reproduced in the *Commentationes Arithmeticæ collectæ*, *t.* II, page 488. He gives formulæ from which may be obtained an indefinite number of triangles answering the conditions of the problem. He deduces, among others, the solution with the integers 68, 85, 87 for the sides. See pages 94-95, of MONTHLY, Vol. IV, 1897, for solution I, by M. Tesch. EDITOR.]

PROBLEMS FOR SOLUTION.

ARITHMETIC.

90. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find the greatest number of inch balls that can be placed in a box 10 inches square and 5 inches deep.

91. Proposed by RAYMOND D. SMITH, Tiffin, Ohio.

A barn 20 feet square is standing in a pasture, and a horse is tied to one corner of it with a rope 50 feet long. Over how much land can he graze ?

92. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pa.

What rate of income do I realize by purchasing United States 4% bonds at 105 if I sell them in six years at 104 ?

*** Solutions of these problems should be sent to B. F. Finkel, not later than April 10.

GEOMETRY.

88. Proposed by FREDERICK R. HONEY, Ph. B., Instructor in Mathematics in Trinity College, New Haven, Conn.

Prove that the volume of the frustum of a cone is equal to one-sixth of the altitude multiplied by the sum of the areas of the upper base, the lower base, and four times the area of the section midway between the upper and lower bases.

89. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Describe a circle tangent to three given circles. [From *Chauvenett's Geometry*, page 318, ex. 213.]

90. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

The bisectors of the angles of the opposite sides (produced) of an inscribed quadrilateral cut the sides at the angular points of a rhombus.

*** Solutions of these problems should be sent to B. F. Finkel, not later than April 10.

CALCULUS.

70. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Graduate Fellow in Mathematics in Vanderbilt University, P. O., Lynnville, Tenn.

$$\text{Prove } \int_0^\infty \frac{\cos ax}{1+x^{2n}}dx = -i\frac{\pi}{n}\sum_{r=1}^{n} \omega^{2r-1}e^{ai\omega^{2r-1}}$$

where n is an integer, a is positive, and ω is $e^{i\pi/2n}$

Is this correct ? Forsyth gives, on page 41, of his *Theory of Functions*, the integral

$$\int_{-\infty}^\infty \frac{\cos axdx}{1+x^{2n}} = -i\frac{\pi}{2l}\sum_{r=1}^{n} \omega^{2r-1}e^{ai\omega^{2r-1}}$$

71. **Proposed by J. C. CORBIN, Pine Bluff, Ark.**

Form the differential equation of the third order, of which $y = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^x$ is the complete primitive.

72. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

A man has a park in the form of a parabolic segment cut off by a chord making an angle $\pi/4$ with the axis. Within the park is a right angled triangular flower plat with one vertex at the center of gravity of the segment, the other vertex at the lower extremity of the chord, and the right angle on the diameter bisecting the chord. The park contains 30 acres, and the perimeter of triangle in linear measure equals the area in square measure. Find the length of the chord, the latus-rectum of the parabola, and the dimensions of the triangle.

*** Solutions of these problems should be sent to J. M. Colaw, not later than April 10.

MECHANICS.

64. **Proposed by B. F. FINKLE, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

A cylindrical vessel, radius of vessel r and altitude h, is filled with water and rests on a horizontal plane. It is required to ascertain the maximum angle of elevation to which the plane may be raised without the vessel falling, allowing the coefficient of friction to be such as to prevent sliding, and the water to overflow as the plane is raised.

65. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

The distance, parallel to the axis, from the mid-point of a chord to the arc of a parabola is constant. Show that the center of gravity of all segments formed by the chord is an equal parabola.

*** Solutions of these problems should be sent to B. F. Finkel, not later than April 10.

DIOPHANTINE ANALYSIS.

62. **Proposed by JOHN M. ARNOLD, Crompton, R. I.**

Find, if possible, four square numbers in arithmetical progression.

63. **Proposed by A. H. HOLMES, Brunswick, Me.**

Given $x^2 + y^3 = 20^3 \times 105498$, to find four positive *integral* values each for x and y.

64. **Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.**

It is required to take from the proper *key* suitable material and hastily construct a "nest" of 10 or 15 prime, integral, rational trapeziums, each containing an area equal to the square root of the product of its four sides.

65. **Proposed by MANSFIELD MERRIMAN, Professor of Civil Engineering, Lehigh University, South Bethlehem, Pa.**

Show that the number 1521 can be expressed in seven different ways as the sum of three perfect squares? Can more than seven different ways be found?

*** Solutions of these problems should be sent to J. M. Colaw, not later than April 10.

AVERAGE AND PROBABILITY.

61. Proposed by COL. CLARKE.

A cube being cut at random by a plane, what is the chance that the section is a hexagon? [From *Williamson's Integral Calculus.*]

62. Proposed by O. S. KIBLER, Superintendent of Schools, Middleburg, O.

A bag contains any number of balls, which are equally likely to be white or black; one is drawn and found to be white. Show that the chance of drawing another white one, the first ball not being replaced, is two-thirds. [From *C. Smith's Treatise on Algebra*, page 615.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than April 10.

MISCELLANEOUS.

60. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

When the Sun's declination is 23° 27′ 15′′ North$=\delta$, in what latitude will it shine on the *north* side of buildings during the first half of the forenoon, and on the *south* side during the other half, and what will be the length of the day ?

61. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

The product of n numbers, each the sum of four squares, may be expressed as the sum of four squares in $(48)^{n-1}$ different ways.

*** Solutions of these problems should be sent to J. M. Colaw, not later than April 10.

NOTES.

THE EVANSTON MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The second meeting of the Chicago Section of the American Mathematical Society was held at Evanston, Ill., on December 30th and 31st, 1897. The first session was called to order in the morning of December 30th by the chairman of the section, Head Professor Moore, University of Chicago. The following list of papers was announced :

1. Independent computation of integrals involving the square root of a quadric or cubic expression. Professor Henry Benner, Albion College.
2. Upon a ruled surface of the fourth order mechanically generated. Dr. E. M. Blake, Purdue University.
3. Upon articulated systems. Dr. E. M. Blake, Purdue University.
4. On the cubic involution and the theory of elliptic functions. Professor Oskar Bolza, University of Chicago.
5. Approximate solution of a particular differential equation. Dr. James H. Boyd, University of Chicago.
6. A note on reticulations. Professor Ellery W. Davis, University of Nebraska.

7. On systems of curves depending upon a single parameter. Dr. L. W. Dowling, University of Wisconsin.
8. A generalized Legendre's coefficient. Dr. James W. Glover, University of Michigan.
9. On a geometrical papyrus of the first century. Mr. Edgar J. Goodspeed, University of Chicago.
10. Alternate processes. Professor Arthur S. Hathaway, Rose Polytechnic Institute.
11. On a remarkable class of hyperspherical tetrahedra. Professor C. H. Hinton, University of Minnesota.
12. Multiple totients. Mr. D. N. Lehmer, University of Nebraska.
13. The determination of all ternary and quaternary symmetrical and alternating collineation-groups. Professor H. Maschke, University of Chicago.
14. On the solvability of groups. Dr. G. A. Miller, Cornell University.
15. Concerning the general equations of the seventh and eighth degrees. Professor E. H. Moore, University of Chicago.
16. Continuous Groups of Spherical Transformations. Professor H. B. Newson, University of Kansas.
17. Normal forms of projective transformations. Professor H. B. Newson, University of Kansas.
18. Cantor's transfinite numbers. Professor James Byrnie Shaw, Illinois College.
19. A new kind of number. Professor James Byrnie Shaw, Illinois College.
20. Twisted quartic curves of the first species and certain associated quartics. Professor Henry S. White, Northwestern University.

Professor Moore stated that a paper from Dr. Dickson, California University, reached him too late to be placed in the list. On account of the richness of the list of papers it was necessary to request the authors to be as brief as possible in presenting them. Regardless of these efforts to save time, the sessions had to be very long in order to get through with the work in two days.

As a whole the papers were of a high grade and furnished another index that we are moving rapidly towards higher types of mathematical activity. It is to be hoped that this forward movement will keep pace with the improvements in our library facilities. Through the generosity of some of our largest libraries it is becoming possible to keep in fairly close touch with the progress in our science, even if one is removed from the library centers.

One of the most pleasant features of the meeting was the fact that nearly all the authors of the papers were present and could thus present their own papers. It was also encouraging to see a number of mathematicians from a distance who were not represented on the program. Among these we may mention Professor Ziwet, editor of the "Bulletin" of the society; Professor Markley, Michigan University; Professor Skinner, Wisconsis University; Professor Waldo, DePaw University; Mr. A. C. Burnham, Illinois University; Professor Winston, Kansas Agricultural College, etc.

In our country of great distances such meetings seem especially important, not only on account of the inspiration resulting from the close contact with those of similar pursuits, but also on account of the opportunities to correct erroneous impressions in regard to the relative importance of the various lines of investigations. It is to be hoped that the future meetings of the Chicago Section will exert a still stronger influence on Western mathematics in both of these lines.

G. A. MILLER.

Ithaca, N. Y. January 5, 1898

EDITORIALS.

Dr. Lovett's article on Lie's Transformation Groups came too late for publication in this issue.

Through the courtesy of T. J. McCormack, assistant editor of the Open Court, we were enabled to present in the January number, a portrait of Leonhard Euler. A portait and biography of Euler appeared in the November number of the Open Court.

We note with pleasure that our valued contributor Dr. G. A. Miller has been appointed instructor in mathematics at Cornell University. His work began the first of this year. The *Cornell Era* of Feb. 5th expresses high appreciation of the fact that Cornell has been fortunate enough to secure so valuable an addition to its Faculty of mathematical instructors. Dr. Miller is a young mathematician of great promise. In the summer of 1895 he went to Germany and spent one year almost entirely in working with Professor Lie, the following year he spent at Paris working with Professor Jordan. That Dr. Miller has done some very fine work in the subject of groups, is sufficiently attested by the fact that both Jordan and Picard have presented his communications to the Paris Academy of Science.

BOOKS AND PERIODICALS.

College Algebra. By Edward A. Bowser, LL. D., Professor of Mathematics and Engineering in Rutgers College, New Brunswick, N. J. 558 pages. Boston: D. C. Heath & Co.

This work has become so far established in favor that it needs no special commendation from us. In matters, arrangement, and manner of treatment this book has numerous good features. It has just been adopted as the text to be used in the new Cosmopolitan University correspondence course. J. M. C.

(1) *Elements of Calculus.* By James M. Taylor, A. M., Professor of Mathematics, Colgate University. 249 pages. Boston: Ginn & Company.

(2) *Elements of Determinants.* By Paul H. Hanus, Assistant Professor of

History and the Art of Teaching, Harvard University. 217 pages. Boston: Ginn & Company.

(1) Prof. Taylor's Calculus is a deservedly popular text-book. By reason of the many illustrations of the elementary processes of the Calculus, it is admirably adapted to the needs of those commencing the subject. Throughout the book many practical problems are given, which serve to exhibit the power and use of the science, and to arouse and keep alive the interest of the student. *Prof. Taylor's Calculus* has also been selected as the text in the Cosmopolitan University Course.

(2) This well-known work, while suited to the needs of the class-room, is especially adapted to self-instruction. The first presentation of the subject is made with great simplicity, but as the student advances less attention is given to details. While the treatise is not voluminous, yet enough is given to show something of the power and utility of determinants and the consequent importance of the study. J. M. C.

Exercises in Choice and Chance. By William Allen Whitworth, M. A., Late Fellow of St. John's College, Cambridge. Price, 6s. 1897. Cambridge: Deighton, Bell & Co.

Prof. Whitworth's book of 700 exercises includes hints for the solution of all the questions in his well-known work on "Choice and Chance," with introductory chapters on the Summation of Certain Series, and a Greshan Lecture on Applications of the Laws of Chance. There are interesting notes on many of the solutions, and the collection of exercises illustrates nearly all the principles and methods arising in questions in probability. Those of our readers who are acquainted with the author's charming little treatise on "Choice and Chance" will note the appearance of the book under review with great satisfaction. J. M. C.

Through Quadratico Equations. By Jos. V. Collins, Ph. D., Professor of of Mathematics, in State Normal School, Stevens Point, Wis. 8vo. Cloth. 85+41 pages. Chicago: Scotts, Foresman & Company.

In the publication of this Manual, the author has availed himself of the opportunity to present in an admirable way some suggestions for the study and teaching of Algebra. Dr. Collins Text-book of Algebra has been adopted in the State of Kansas and it was due to this fact that the publication of the Manual was made necessary. The Manual contains many historical notes of great interest in addition to much original matter. Dr. Collins is a strong advocate of the disuse of the cumbersome radical sign. See his article in MONTHLY, Vol. II, No. 4. B. F. F.

Analytic Functions. Suitable to Represent Substitutions. By Leonard E. Dickson, Ph. D. Quarto Pamphlet, 10 pages.

The above is a reprint from the American Journal of Mathematics and was written while Dr. Dickson was pursueing his coarse of mathematics in the University of Chicago.

The Analytic Representation of Substitutions on a Power of Prime Number of Letters with a discussion of the Linear Group. A dissertation presented to the faculty of Arts, Literature and Science, of the University of Chicago for the degree of Doctor of Philosophy. By Leonard Eugene Dickson.

This dissertation is on a subject in which Dr. Dickson is a recognized authority, both in America and Europe. In this thesis he has generalized the results in his article referred to above, and in this wholy original work Dr. Dickson has earned with great credit the honor that the University has conferred upon him, and that at an exceedingly early age, he being no older than twenty-two at the time he received his degree.

On Rational Quadratic Transformations. By H. W. Haskell, Ph. D., Associate Professor of Mathematics, University of California.

This paper is contained in the Proceedings of the California Academy of Sciences, February, 1898. The Quadratic Cremona Transformation. By L. E. Dickson, Ph. D., Instructor in Mathematics, University of California.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

The March number of the *American Monthly Review of Reviews* is another achievement in monthly journalism. The topics treated in this magazine are such as occupy much space in the daily press, but the *Review* is able to treat them more deliberately and in a more carefully adjusted proportion. No other illustrated monthly appearing on the first day of March will have so much as a reference to the De Lome letter, the *Maine* disaster, or the Zola trial in Paris; but these great themes of the hour are fully discussed in the *Review's* pages. The *Review's* readers expect to have them discussed there, so accustomed have they become to the essential qualities of timeliness and comprehensiveness in the "busy man's magazine."

The Open Court. A Monthly Magazine. Devoted to Science of Religion, the Religion of Science, and the extension of the Religious Parlament Idea. Edited by Dr. Paul Carus; Assistant Editor, T. J. McCormack and Associate Editors E. C. Hegeler and Mary Carus. Price $1.00 per year in advance. Single copies, 10 cents.

The December number (1897) contains biography of Lagrange, by Assistant Editor T. J. McCormack. The Frontispiece of this number contains an excellent portrait of that master mathematician. The Jan. number contains a biography and portrait of Laplace. Mr. McCormack has gone to a great deal of trouble and expense in securing portraits of the great masters in mathematics, and it is probable that he can furnish, at a reasonable price, the portraits of most of these great men to any of our readers who may desire them. Other biographies and portraits will appear in future numbers of the *Open Court.*

SOME ERRATA IN JANUARY NUMBER.

Page 14, line 11, for "$(2m-)$" read $(2m-1)$.

Page 15, line 9, for "6/7" read 7/6.

Page 17, line 13, for "Hewlett" read Woodmere; line 15, for "forms" read form; line 21, for "0.04−" read 0.04+; line 22, for "+.13" read 8.13, and for "error+*" read error*; line 25, for "−2.0316+" read −3.0416+.

Page 22, line 20, supply .04 so as to read −3.04; line 22, for "−3.−04, +3.04.

Page 25, line 14, in numerator, for "$n^2[x(n^2-1)+1p]$" read $n^2[q(n^2-1)+2p]$; line 17, for "Then" read Take.

Page 27, line 21, for "horison" read horizon; in figure, join BM', for "S" at right read S' and supply M in $M'MM''$.

Page 28, line 6, for "81° 36° 29''" read 81° 36′ 29″; the figure should be drawn so that ACB pass through M, and E should be on RFT.

Page 29, problem 68, line 1, x should stand in index position with respect to a, and in line 2, the small plus signs should be raised to intermediate position, and where "$k-1-1$" occurs, last -1 should be lowered to line of these signs; line 17, for "State University" read "University of Oregon."

Page 30, line 1 of No. 60, for "three" read six; line 2 of No. 60, for "five times" read five-half times, and for "three" read six.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. MARCH, 1898. NO. 3.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from January Number.]

PROPOSITION XXXII. *Now I say there is (in hypothesis of acute angle) a certain determinate acute angle BAX drawn under which AX* (Fig. 33.) *only at an infinite distance meets BX, and thus is a limit in part from within, in part from without; on the one hand of all those which under lesser acute angles meet the aforesaid BX at a finite distance; on the other hand also of the others which under greater acute angles, even to a right angle inclusive, have a common perpendicular in two distinct points with BX.*

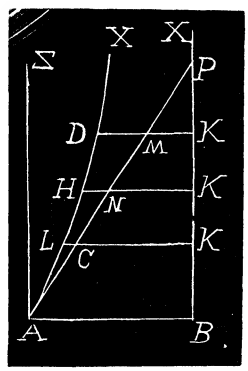


Fig. 33.

PROOF. First it holds (from Cor. II. after Proposition XXIX.) that no determinate acute angle will be the greatest of all drawn under which a straight from the point A meets the aforesaid BX at a finite distance

Secondly, it holds in like manner that (in hypothesis of acute angle) no acute angle will be the least of all drawn under which a straight has a common perpendicular in two distinct points with BX; since indeed (from what precedes) there can be no determinate limit, such that there cannot be found, under a lesser angle

constituted at the point A, a common perpendicular in two distinct points, which is less than any assignable length R.

And hence follows thirdly, that (in this hypothesis) there must be a certain determinate acute angle BAX, drawn under which AX so approaches ever more to BX, that only at an infinite distance does it meet it.

But further that this AX is a limit in part from within in part from without of each of the aforesaid classes of straights is proved thus. First, it agrees with those straights which meet BX at a finite distance since it also finally meets; but it differs, because it meets only at an infinite distance.

But secondly it also agrees, and at the same time differs from those straights which have a common perpendicular in two distinct points with BX; because it also has a common perpendicular with BX; but in one and the same point X infinitely distant. But this latter ought to be considered demonstrated in Proposition XXVIII., as I point out in its corollary.

Therefore it holds, that (in the hypothesis of acute angle) there will be a certain determinate acute angle BAX, drawn under which AX only at an infinite distance meets BX, and thus is a limit in part from within, in part from without; on the one hand of all those which under lesser acute angles meet the aforesaid BX at a finite distance; on the other hand also of the others which under greater acute angles, even to a right angle inclusive, have a common perpendicular in two distinct points with BX. Quod erat etc.

[To be Continued.]

ON THE BEST METHOD OF SOLVING THE MARKINGS OF JUDGES OF CONTESTS.

By F. R. MOULTON.

The fact that many different methods are in use for deciding from the markings of judges the relative standing of the participants in oratorical and similar contests, and that several different methods have been stated to be the best by Professors of Mathematics in some of our colleges, may be taken as the excuse for this paper. It is questionable whether such a problem can be solved by perfectly rigorous mathematical processes, but it has seemed that a method similar to that of Hagen in the theory of probability may be applied with advantage. Let us agree to adopt the following hypotheses.

(1) The judges mark each contestant independently and by the same scale.

(2) There is a true marking for each contestant as compared to a fixed ideal.

(3) The deviation of each judge's markings from these true markings, are the result of a very great many influences, such as, the inherited inclinations re-

sulting from the entire experience of the judge's ancestry, the judge's own experiences, his business or profession, health, etc., etc.

These influences will have different degrees of importance. Let us select a unit in terms of which all the influences acting upon every judge can be expressed. Then we shall consider that an influence whose importance is k times the unit influence is equivalent to k unit influences working together. With this understanding the whole number of influences is some large number, μ. The coefficients of those which do not operate in the case of particular judges are zero.

(4) In the long run the unit influence will be just as liable to cause the judge to err in one direction as the other. (This is plainly a pure assumption but is the only one that can be made.)

Let us now consider the question in two parts ; first, an investigation of the law of distribution of errors, and second, the best way of eliminating their influence.

I. *Investigation of the expression for the probability of the existence of an error of a given magnitude.*

By hypothesis (4) the probability that a unit influence will give a positive error is $p=\frac{1}{2}$, and negative $q=\frac{1}{2}$.

Then all possible combinations of positive and negative errors is given by the terms of the expansion

$$(1) \qquad (p+q)^{\mu}=1=\Sigma\frac{\mu\,!}{m\,!\,n\,!}p^{m}q^{n}$$

where $m+n=\mu$. The general term is

$$(2) \qquad T_n=\frac{\mu\,!}{m\,!\,n\,!}p^{m}q^{n}$$

which is the probability of m positive and n negative errors. By Stirling's theorem

$$\mu\,!=\mu^{\mu}e^{-\mu}\sqrt{2\pi\mu}.$$

Therefore (2) becomes

$$(3) \qquad T_n=\left(\frac{\mu p}{m}\right)^{m}\left(\frac{\mu q}{n}\right)^{n}\frac{\sqrt{\mu}}{\sqrt{(2\pi\mu)}}=\left(\frac{\mu p}{m}\right)^{m+\frac{1}{2}}\left(\frac{\mu q}{n}\right)^{n+\frac{1}{2}}\frac{1}{\sqrt{(2\pi\mu pq)}}.$$

We now find what values of m and n make T_n a maximum. Changing n to $n-1$ and $n+1$ successively in (2) we have

$$(4) \qquad \begin{cases} T_{n-1}=\dfrac{\mu\,!}{(m+1)!\,(n-1)!}p^{m+1}q^{n-1}=T_n.\dfrac{n}{m+1}\dfrac{p}{q} \\[2ex] T_{n+1}=\dfrac{\mu\,!}{(m-1)!\,(n+1)!}p^{m-1}q^{n+1}=T_n.\dfrac{m}{n+1}\dfrac{q}{p} \end{cases}$$

If T_n is a maximum $T_n>T_{n\pm1}$. Therefore

$$(5)\qquad \begin{cases} 1 > \dfrac{n}{m+1}\,\dfrac{p}{q} \text{ and } 1 > \dfrac{m}{n+1}\,\dfrac{q}{p} \\ m+n=\mu \text{ and } q=1-p. \end{cases}$$

Eliminating n and q we have

$$1 > \frac{\mu-m}{m+1}\,\frac{p}{1-p} \text{ and } 1 > \frac{m}{\mu-m+1}\,\frac{1-p}{p}.$$

This may be written

$$m/\mu > p + \frac{m}{\mu(\mu+1)} - \frac{1}{\mu+1} \text{ and } m/\mu < p + \frac{m}{\mu(\mu+1)}.$$

Hence as μ increases we have more and more nearly $m=\mu p$ and similarly $n=\mu q$ or $(\mu p/m)=1$ and $(\mu q/n)=1$.

Substituting these values in (3) we have

$$T_n(max) \doteq G = \frac{1}{\sqrt{2\pi\mu pq}} \text{ and}$$

$$(6)\qquad T_n = G\left(\frac{\mu p}{m}\right)^{m+\frac{1}{2}}\left(\frac{\mu q}{n}\right)^{n+\frac{1}{2}}$$

Let $G_{n\pm l}$ be the value which $T_{n\pm l}$ takes for $m=\mu p$; $n=\mu q$. Therefore,

$$(7)\qquad G_{n-l} = G[1+(l/m)]^{-m-l-\frac{1}{2}}[1-(l/n)]^{-n+l-\frac{1}{2}}.$$

Since we always have $|\,l\,| < m$ we have

$$[1+(l/m)]^{-m-l-\frac{1}{2}} = e^{(-m-l-\frac{1}{2})\log(1+l/m)} = e^{[(-m-l-\frac{1}{2}),m]+[(ml^2+l^3+\frac{1}{2}l^2)/(2m^2)]\ldots\ldots}$$

Substituting in (7) we obtain

$$(8)\qquad \begin{cases} G_{n-l} = G.e^{\left\{\frac{-l^2+l}{2m}+\frac{l^3+\frac{3}{2}l^2}{m^2}\cdots\cdots\frac{-l^2-l}{2n}+\frac{-l-\frac{3}{2}l^2}{6n^2}\cdots\cdots\right\}} \\ G_{n+l} = G.e^{\left\{\frac{-l^2-l}{2m}+\frac{-l^3-\frac{3}{2}l^2}{6m^2}\cdots\cdots\frac{-l^2-l}{2n}+\frac{l^3+\frac{3}{2}l^2}{6n^2}\cdots\cdots\right\}} \end{cases}$$

The number of positive errors in G_{n-l} is

$$(9)\qquad \begin{cases} \mu p+l \text{ and negative } \mu q-l. \text{ In } G_{n+l} \text{ the positive are} \\ \mu p-l \text{ and negative } \mu q+l. \end{cases}$$

By our hypothesis (4) $p=q=\frac{1}{2}$. Then we get from (8), neglecting terms higher than the first order in $1/\mu$.

(10) $$G_{n-l}=G_{n+l}=Ge^{-l^2/\frac{1}{2}\mu}.$$

Suppose that the absolute value of the error produced by a unit influence is α, then the errors in this case are by (9),

(11) $$\begin{cases}(\frac{1}{2}\mu+l)\alpha-(\frac{1}{2}\mu-l)\alpha=2l\alpha=x\\(\frac{1}{2}\mu-l)\alpha-(\frac{1}{2}\mu+l)\alpha=-2l\alpha=-x.\end{cases}$$

Increasing l by unity increases the right side by $2\alpha=\triangle x$, a very small quantity from the nature of the problem. From $2\alpha=\triangle x$ and (11) we have

(12) $$l=x/\triangle x.$$

Hence (10) becomes

(13) $$G_{n-l}=G_{n+l}=Ge^{-\{x^2/[(\frac{1}{2}\mu)\triangle x^2]\}}$$

We may, without loss of generality, suppose α and consequently x to become indefinitely small, and at the same time μ will become indefinitely great. The absolute value of one unit error is $\frac{1}{2}\triangle x$, and if they are all of the same character their sum, $\mu(\triangle x/2)$, which is the maximum error possible, becomes from the nature of the problem an indefinitely large quantity. Hence $(\frac{1}{2}\mu)(\triangle x)^2$ equals a finite constant, say, $1/h^2$. Then we have

(14) $$G=(h/\sqrt{\pi})dx$$

when $\triangle x$ becomes dx. This is the probability of the error zero. Substituting in (13) we have

(15) $$G_{n-l}=G_{n+l}=(h/\sqrt{\pi})e^{-h^2x^2}dx=px,$$

which is the probability of an error with the magnitude x. This is the well-known probability formula.

II. *The method of obtaining the most probable standing of each contestant.*

Let $w_1, w_2 \ldots\ldots w_n$ be the markings given to a contestant by the respective judges. Let z be the true marking which the contestant deserves.

Let $z-w_1=\varepsilon_1$; $z-w_2=\varepsilon_2$ $\ldots\ldots z-w_n=\varepsilon_n$. Then by (15) the probability that the error ε_i will occur is

$$p\varepsilon_i=\frac{h}{\sqrt{\pi}}e^{-h^2\varepsilon_i^2}d\varepsilon_i.$$

The probability that the errors $\varepsilon_1, \varepsilon_2 \ldots\ldots \varepsilon_n$ will occur together in one marking of each judge is,

(16) $$P=p\varepsilon_1.p\varepsilon_2\ldots\ldots p\varepsilon_n=\frac{h^n}{(\sqrt{\pi})^n}e^{-h^2(\varepsilon_1^2+\varepsilon_2^2+\ldots+\varepsilon_n^2)}d\varepsilon_1 d\varepsilon_2\ldots d\varepsilon_n$$

The most probable value of z is that one which will make P a maximum. P is a maximum when $(\varepsilon_1^2+\varepsilon_2^2+\ldots\ldots\varepsilon_n^2)$ is a minimum; or, by equating to zero, the first derivative, when $\varepsilon_1+\varepsilon_2+\ldots\ldots\varepsilon_n=0$. That is when $z-w_1+z-w_2+\ldots\ldots z-w_n=0$. Solving for z we find

(17) $$z=(w_1+w_2+\ldots\ldots w_n)/n.$$

Hence it appears that in the long run the best way to solve the markings of judges of contests is simply to take the arithmetical means of the markings when the merits of the contestants are compared with ideal standards.

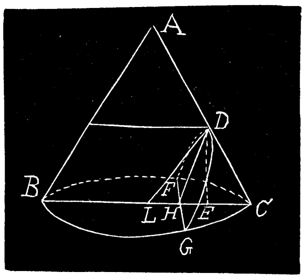
The methods for successive approximations have already been discussed in this journal.

The University of Chicago.

A METHOD OF DEFINING THE ELLIPSE, HYPERBOLA AND PARABOLA AS CONIC SECTIONS.

. By W. W. LANDIS, A. M., Professor of Mathematics in Dickinson College, Carlisle, Pa.

ABC is a right circular cone, the angle at the vertex being 2α. DFG is a plane section making an angle θ with the axis of the cone. Take D as the origin, DH as the axis of x, and a perpendicular through D as the axis of y. We seek to find a relation between x and y, the parameters being $g(=AD)$ and α, and the variable parameter θ. In the circle BGC, $y^2=(a\pm b)c=ac\pm bc\ldots\ldots(1)$, where $a=BL$, $b=LH$, and $c=HC$. In the isosceles triangle DLC, $x^2=d^2\mp bc\ldots\ldots(2)$, where $d=DC=DL$. Adding (1) and (2), $x^2+y^2=ac+d^2\ldots\ldots(3)$.



Now $c=x\sin\theta+d\sin\alpha$,

$d=x\cos\theta\sec\alpha=x\cos\theta/\cos\alpha$, and

$a=2g\sin\alpha$.

Making this substitution we get

$$x^2+y^2=x^2\frac{\cos^2\theta}{\cos^2\alpha}+2g\sin\alpha\left[x\sin\theta+x\frac{\cos\theta\sin\alpha}{\cos\alpha}\right]$$

$$\text{or } x^2\left[1-\frac{\cos^2\theta}{\cos^2\alpha}\right]+y^2-2gx\sin\alpha[\sin\theta+\cos\theta\tan\alpha]=0\ldots\ldots(4)$$

which we may write

$$x^2\left[\frac{\sin^2\theta-\sin^2\alpha}{1-\sin^2\alpha}\right]+y^2-2gx\sin\alpha[\sin\theta+\cos\theta\tan\alpha]=0\ldots\ldots(5).$$

Now if $\theta=\alpha$, the plane then being parallel to one and only one element, (5) reduces to $y^2-4gx\sin^2\alpha=0$, a parabola of latus rectum $=4g\sin^2\alpha$.

If $\theta>\alpha$ the section cuts all elements and the coefficients of x^2 and y^2 are both positive, and we have an ellipse whose center, axes, and eccentricity are readily found; and in particular if $\theta=90^\circ$, the section is parallel to the base, the coefficients of x^2 and y^2 are unity and we have a circle, whose center is $(g\sin\alpha, 0)$. If $\theta<\alpha$, the section cuts both nappes, the coefficients of x^2 and y^2 are of unlike sign and we have a hyperbola.

If $g=0$, (5) becomes $y=\pm x\sqrt{\dfrac{\sin^2\alpha-\sin^2\theta}{\cos^2\alpha}}$....(6). Now if $\theta=\alpha$, (6) becomes $y=0$, a straight line, the limit of the parabola. If $\theta<\alpha$, (6) represents two real straight lines, the limiting case of the hyperbola. And if $\theta>\alpha$, (6) represents two imaginary lines intersecting in the real point (0, 0), which is the limiting form of the ellipse.

The equations (5) and (6) show the dependence of the nature of the conic sections upon the angle which the cutting plane makes with the axis, and the dependence of their shape upon the angle of the cone and the distance from the vertex to the first element cut.

REMARK 1. If the section be a parabola, the foot of the perpendicular from the middle point of the line through D parallel to BC, upon DH, is the focus.

REMARK 2. The eccentricity of any conic section is $\varepsilon=[\cos\theta/\cos\alpha]$.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from November Number.]

LXX. Fig. 31.

$AEM=ACB$ of $AEHC$.

$MOL=ACB=ADK+DKBC=BHI+DKBC$. $LOI=BEK$.

$\therefore ABLM \asymp BCDF+AEHC$.

LXXI. Fig. 31.

$AMPQ \asymp AMOC \asymp AEHC$. $BLPQ \asymp BLOC \asymp BCDF$.

$\therefore ABLM \asymp BCDF+AEHC$.

LXXII. Fig. 31.

$MTE=BSF$. $\therefore BS=MT$. $\therefore AMLB \asymp 2AMTS$.

But $AMTS \asymp EAC+FBC$; since $MTE=BSF$, and $AEM=ACB$.

$\therefore 2AMTS \asymp 2EAC+2FBC \asymp AEHC+BCDF$.

$\therefore AMLB \asymp BCDF+AEHC$.

LXXIII. Fig. 31.

$ABVU \simeq ABHW$. $UVLM \simeq 2MLH \simeq 2MO\mathrm{L}+2\mathrm{L}OH \simeq HEW+ACB+2FBC(\simeq BCDF)$.

$\therefore ABLM \simeq BCDF+AEHC$.

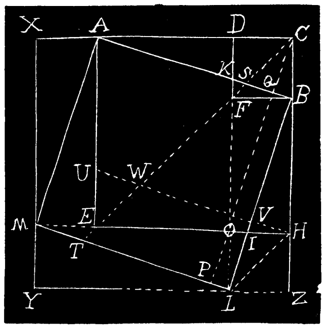


Fig. 31.

LXXIV. Fig. 31.

$ABLM \simeq CZYX-4ACB$.

$BCDF+AEHC=0\mathrm{LZ}H+AEHC \simeq CZYX-2MO\mathrm{L}Y \simeq CZYX-4ACB$.

$\therefore ABLM \simeq BCDF+AEHC$.

NOTE.—This proof is similar to that of Henry Boad's, London, 1733.

LXXV. Fig. 32.

$ANML \simeq 2ACL=2AFB \simeq AFHC$.

$NMKB \simeq 2BCK \simeq BCDE$.

$\therefore ABKL \simeq BCDE+AFHC$. *Wipper.*

LXXVI. Fig. 32.

Since EC produced to O passes through the center of $ABKL$, $ABQO \simeq \frac{1}{4}ABKL$. Now, $APCO \simeq CAF$, since $ACO=AFP$; $PBQC \simeq CBE$, since $BCP=BEQ$. $\therefore ABKL \simeq BCDE+AFHC$.

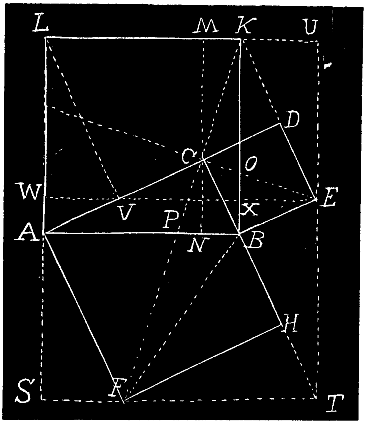


Fig. 32.

LXXVII. Fig. 32.

$LVDK=AFHB$. Then, $AVL=BEK$.

$\therefore ABKL \simeq BCDE+AFHC$.

LXXVIII. Fig. 32.

$ABKL \simeq STU\mathrm{L}-2ASF-3FHT-AFHB$.

$BCDE+AFHC \simeq STUL-2ASF-3FHT-LVDK(=AFHB)$.

$\therefore ABKL \simeq BCDE+AFHC$.

LXXIX. Fig. 32.

$ABXW \simeq ABEV \simeq BCDE$.

$WXK\mathrm{L} \simeq VEKL \simeq AFHC$, since $VED=ABC$, and $VDKL=FHBA$.

$\therefore ABKL \simeq BCDE+AFHC$.

LXXX. Fig. 33.

$AOML \simeq ANCL \simeq 2ANC \simeq AFHC$.

$OBKM \simeq 2BCK \simeq BCDE$, since SK, the altitude of BCK, $=BC$.

$\therefore ABKL \simeq BCDE+AFHC$.

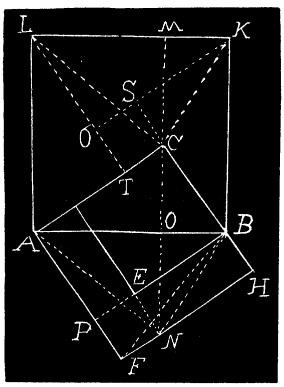


Fig. 33.

LXXXI. Fig. 33.

$AOML \simeq 2ACL=2AFB \simeq AFHC$.

$OBKM \simeq BKCN \simeq 2BCN \simeq BCDE$.

$\therefore ABKL \simeq BCDE+AFHC$.

LXXXII. Fig. 33.

$QTCS=PFNE$.

$ATL+LQK+KSB=APB+CDN+NHC$.

$\therefore ABKL \simeq BCDE+AFHC$.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton, New Jersey.

V.

INTRODUCTION OF NEW VARIABLES IN A ONE PARAMETER GROUP. LIE'S SYMBOL OF AN INFINITESIMAL TRANSFORMATION. CANONICAL FORM AND CANONICAL VARIABLES OF A ONE PARAMETER GROUP.

1. A point transformation has been defined to be an operation by which a point (x, y) is carried to the position of some point (x_1, y_1). If this operation is applied to all the points of the xy-plane simultaneously, we have a point transformation of the plane into itself. Such an operation has been represented analytically by two equations of the form

$$x_1=\varphi(x, y), \quad y_1=\psi(x, y), \tag{1}$$

where x, y represent the original points and x_1, y_1 their new positions after the transformation has been effected, and where finally the fractions φ and ψ are independent analytical functions. By introducing a parameter a into the equations (1) they will no longer represent a single transformation but an infinite number of transformations forming a family of ∞^1 transformations given by

$$x_1=\varphi(x, y, a), \quad y_1=\psi(x, y, a). \tag{2}$$

If now this family contains the inverse transformation of every transformation in it and the family possesses the property that the successive performance or product of any two transformations of the family is equivalent to a single transformation belonging to the family, the equations (2) are said to define a one parameter, finite, continuous group of transformations.

The reader will observe that the transformation (1) could have been represented by the equations

$$r_1=\rho(r, \theta), \quad \theta_1=\sigma(r, \theta), \tag{3}$$

where r, θ are the polar coördinates of the initial position of the point and r_1, θ_1 the coördinates of its final position, the functions ρ and σ being independent analytical functions. Similarly the one parameter group of transformations (2) could have been represented in polar coördinates by the two equations

$$r_1=\rho(r, \theta, \alpha), \quad \theta_1=\sigma(r, \theta, \alpha), \tag{4}$$

where α is a parameter. Hence we see that the operation (1) and the group property of the family (3) are independent of the coördinate system to which the points of the plane are referred. Accordingly, if in the equations

$$x_1=\varphi(x,\ y,\ t),\quad y_1=\psi(x,\ y,\ t) \tag{5}$$

of a one parameter group new variables $\bar{x}$, $\bar{y}$, $\bar{x}_1$, $\bar{y}_1$ are introduced, where

$$\left.\begin{aligned}\bar{x}&=\lambda(x,\ y), & \bar{y}&=\mu(x,\ y),\\ \bar{x}_1&=\lambda(x_1,\ y_1), & \bar{y}_1&=\mu(x_1,\ y_1),\end{aligned}\right\} \tag{6}$$

the new equations

$$x_1=\Phi(x,\ y,\ t),\quad y_1=\Psi(x,\ y,\ t) \tag{7}$$

obtained from the original ones (3) by these substitutions (6) will also represent a one parameter group.

Thus for example, it is easy to show that the ∞^1 transformations given by the equations

$$x_1=x+t,\quad y_1=\frac{xy}{x+t} \tag{8}$$

represent a one parameter group, for if $(x_1,\ y_1)$ be transformed into $(x_2,\ y_2)$ we have

$$x_2=x_1+t_1,\quad y_2=\frac{x_1y_1}{x_1+t_1}. \tag{9}$$

Eliminating the variables $(x_1,\ y_1)$ from the equations (9) by means of the equations (8) we find

$$x_2=x+(t+t_1),\quad y_2=\frac{xy}{x+(t+t_1)}.$$

These last equations prove that the family (8) is a group, since they represent the product of two transformations of the family and are of the same form as the defining equations (8) of the family.

Now let us introduce new variables x, y into the equations of the group defined by the equations

$$\bar{x}=x/y,\quad \bar{y}=xy; \tag{10}$$

then putting

$$\bar{x}_1=x_1/y_1,\quad \bar{y}_1=x_1y_1, \tag{11}$$

and eliminating x, y, x_1, y_1 from the equations (8), (10) and (11), we have

$$\left.\begin{aligned}\bar{x}_1&=\frac{(x+t)^2}{xy}=\frac{(\sqrt{\bar{x}\bar{y}}+t)^2}{\bar{y}},\\ \bar{y}_1&=xy=\bar{y}.\end{aligned}\right\} \tag{12}$$

The point $(\bar{x},\ \bar{y})$ is carried to the position $(\bar{x}_1,\ \bar{y}_1)$ by the transformation (12) of the family (12), and a transformation corresponding to the parameter t_1 carries $(\bar{x}_1,\ \bar{y}_1)$ to the position $(\bar{x}_2,\ \bar{y}_2)$, say, where

$$\left.\begin{aligned}\bar{y}_2&=\frac{(\sqrt{\bar{x}_1\,,\,\bar{y}_1}+t_1)^2}{\bar{y}_1},\\ \bar{y}_2&=\bar{y}_1.\end{aligned}\right\} \qquad (13)$$

The elimination of $\bar{x}_1$, $\bar{y}_1$ by means of the equations (12) and (13) gives,

$$\begin{aligned}\bar{x}_2&=\frac{(\sqrt{\bar{x}\bar{y}}+t+t_1)^2}{y},\\ \bar{y}_2&=\bar{y},\end{aligned} \qquad (14)$$

the equations of the transformation of the family (12) which carries the point $(\bar{x}, \bar{y})$ directly to the position $(\bar{x}_2, \bar{y}_2)$; the form of these equations shows that this transformation belongs to the family ; hence the family is a group.

2. The reader will recall that we have established in preceding sections LIE's theorem relative to the connection between the notions infinitesimal transformation and one parameter group. This relation proved to be a one-to-one correspondence of such an intimate relation that the infinitesimal transformation of a one parameter group may be taken as the complete representative of the latter without loss of generality or property. In order to make the fullest use of this equivalence of notions LIE has devised a very convenient symbol for an infinitesimal transformation. This symbol is readily constructed as follows :

Let the finite equations of the group be

$$x_1=\varphi(x, y, t), \quad y_1=\psi(x, y, t) ; \qquad (15)$$

and those of its infinitesimal transformation be

$$x'=x_1+\xi(x_1, y_1)\delta t+\ldots\ldots, \quad y'=y_1+\eta(x_1, y_1)\delta t \;\ldots\ldots ; \qquad (16)$$

and let the identical transformation of the group correspond to the zero value of the parameter t.

By virtue of the infinitesimal transformation (16), x_1 and y_1 receive the increments

$$\delta x_1=\xi(x_1, y_1)\delta t, \quad \delta y_1=\eta(x_1, y_1)\delta t. \qquad (17)$$

The increment which an arbitrary function $f(x_1, y_1)$ receives by this infinitesimal transformation is found by substituting these values (17) in

$$\delta f_1=\frac{\partial f(x_1, y_1)}{\partial x_1}\delta x_1 + \frac{\partial f(x_1, y_1)}{\partial y_1}\delta y_1,$$

to be

$$\delta f_1=\{\xi(x_1, y_1)\frac{\partial f(x_1, y_1)}{\partial x_1}+\eta(x_1, y_1)\frac{\partial f(x_1, y_1)}{\partial y_1}\}\delta t ;$$

since $f(x_1, y_1)$ becomes $f(x, y)$ when $t=0$, we have for the increment of the function $f(x, y)$

$$\delta f=\{\xi(x, y)\frac{\partial f}{\partial x}+\eta(x, y)\frac{\partial f}{\partial y}\}\delta t. \tag{18}$$

Conversely, when the increment which an arbitrary function receives by an infinitesimal transformation is known, the infinitesimal transformation itself is known ; for putting $f\equiv x$, (18) becomes

$$\delta t=\xi(x, y)\delta t\ ;$$

and putting $f\equiv y$,

$$\delta t=\eta(x, y)\delta t.$$

Hence instead of defining an infinitesimal transformation by means of two equations (16) it may be characterized by the expression

$$\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y},$$

which is equal to the quotient of the increment of an arbitrary function $f(x, y)$ by the infinitesimal transformation divided by δt. For this reason LIE adopts the symbol

$$Uf\equiv\frac{\delta f}{\delta t}=\xi(x, y)\frac{\partial f}{\partial x}+\eta(x, y)\frac{\partial f}{\partial y} \tag{19}$$

as the symbol of the infinitesimal transformation (16).

If we replace $f(x, y)$ by x in this expression we find $Ux=\xi$, and if we put y for f we have $Uy=\eta$, hence the symbol may be written

$$Uf=Ux\frac{\partial t}{\partial x}+Uy\frac{\partial t}{\partial y}. \tag{20}$$

3. The question arises—How is the form of this symbol (19) affected by the introduction of new variables ? The solution of this question leads to one of the remarkable properties of the symbol.

Let the new variables be

$$\bar{x}=\lambda(x, y),\quad \bar{y}=\mu(\bar{x}, \bar{y})\ ;$$

by their substitution the function $f(x, y)$ becomes $f(\bar{x}, \bar{y})$, and the principles of partial differentiation give

$$\frac{\partial t}{\partial x}=\frac{\partial f}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x}+\frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial x},\qquad \frac{\partial f}{\partial y}=\frac{\partial f}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial y}+\frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial y}.$$

Hence

$$Uf \equiv \xi(x,\ y)\frac{\partial f}{\partial x} + \eta(x,\ y)\frac{\partial f}{\partial y}$$

becomes

$$\xi\left(\frac{\partial t}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} + \frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial x}\right) + \eta\left(\frac{\partial f}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial y} + \frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial y}\right);$$

or

$$\left(\xi\frac{\partial \bar{x}}{\partial x} + \eta\frac{\partial \bar{x}}{\partial y}\right)\frac{\partial f}{\partial \bar{x}} + \left(\xi\frac{\partial \bar{y}}{\partial x} + \eta\frac{\partial \bar{y}}{dy}\right)\frac{\partial f}{\partial \bar{y}};$$

or finally designating Uf in the new variables by $\bar{U}f$,

$$\bar{U}f = U\bar{x}\frac{\partial f}{\partial \bar{x}} + U\bar{y}\frac{\partial f}{\partial \bar{y}}.$$

Hence if new variables, $\bar{x}$, $\bar{y}$, $\bar{x}_1$, $\bar{y}_1$ are introduced into the equations of a one parameter group,

$$x_1 = \varphi(x,\ y,\ t), \quad y_1 = \psi(x,\ y,\ t),$$

by means of the equations,

$$\bar{x} = \lambda(x,\ y), \quad \bar{y} = \mu(x,\ y),$$

$$\bar{x}_1 = \lambda(x,\ y), \quad \bar{y}_1 = \mu(x,\ y),$$

the symbol $\bar{U}f$ of the infinitesimal transformation of the new group can be determined directly by introducing the variables $\bar{x}$, $\bar{y}$ into the symbol Uf of the original group.

4. The result of the preceding paragraph gives rise to the fact that by introducing new variables $\bar{x}$, $\bar{y}$ any given infinitesimal transformation

$$Uf \equiv \xi(x,\ y)\frac{\partial f}{\partial x} + \eta(x,\ y)\frac{\partial f}{\partial y}$$

can be brought to the form of any other

$$\bar{U}f \equiv \bar{\xi}(\bar{x},\ \bar{y})\frac{\partial f}{\partial \bar{x}} + \bar{\eta}(\bar{x},\ \bar{y})\frac{\partial f}{\partial \bar{y}}.$$

In order to make this transformation it is only necessary to choose $\bar{x}$ and $\bar{y}$ as such functions of x, y that the identity

$$\bar{\xi}\frac{\partial f}{\partial \bar{x}} + \bar{\eta}\frac{\partial f}{\partial \bar{y}} = U\bar{x}\frac{\partial f}{\partial \bar{x}} + U\bar{y}\frac{\partial f}{\partial \bar{y}},$$

shall exist for all values of f. Applying the theorem of undetermined coefficients the preceding identity must break up into the two

$$U\bar{x}=\bar{\xi}, \quad U\bar{y}=\bar{\eta},$$

or written in full,

$$\xi\frac{\partial \bar{x}}{\partial x}+\eta\frac{\partial \bar{x}}{\partial y}=\bar{\xi}, \qquad \xi\frac{\partial \bar{y}}{\partial x}+\eta\frac{\partial \bar{y}}{\partial y}=\bar{\eta}.$$

These are two partial differential equations and we know that they have independent solutions if $\bar{\xi}$ and $\bar{\eta}$ are not both identically zero ; hence the proposed transformation is always possible since Uf is the representative of the original group and $\bar{U}f$ that of the transformed group. This proves LIE's theorem :

By introducing new variables every one parameter group of the plane can be changed into every other one parameter group of the same plane.

5. If $\bar{U}f$ has the particular form

$$\bar{U}f\equiv\frac{\partial f}{\partial \bar{y}},$$

we have the remarkable theorem that *every one parameter group of the plane can be brought to the form of a group of translations by a proper change of variables.*

These new variables $\bar{x}$ and $\bar{y}$ are found by integrating the differential equations

$$U\bar{x}\equiv\xi\frac{\partial \bar{x}}{\partial x}+\eta\frac{\partial \bar{x}}{\partial y}=0, \quad U\bar{y}\equiv\xi\frac{\partial \bar{y}}{\partial x}+\eta\frac{\partial \bar{y}}{\partial y}=1, \tag{21}$$

since in the form $\bar{U}f\equiv\frac{\partial f}{\partial \bar{y}}$, $\bar{\xi}=0$, and $\bar{\eta}=1$.

Now if we recall the fact that a partial differential equation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0$$

is equivalent to the ordinary differential equation

$$\frac{dx}{x}=\frac{dy}{y},$$

and that the partial differential equation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=1$$

is equivalent to the simultaneous system

$$\frac{dx}{x}=\frac{dy}{y}=df,$$

the equations (21) can be replaced by the following

$$\frac{dx}{\xi}=\frac{dy}{\eta}, \tag{22}$$

$$\frac{dx}{\xi}=\frac{dy}{\eta}=d\,\bar{y}. \tag{23}$$

The integration of (22) gives the function $\bar{x}$ say. This substituted in (23), the latter can be integrated by a quadrature and determines $\bar{y}$.

That the infinitesimal transformation

$$\bar{U}f\equiv\frac{\partial f}{\partial \bar{y}}$$

generates a group of translations parallel to the $\bar{y}$-axis is easily shown by applying a theorem of Lie proved in a previous paragraph, namely, the theorem which shows how to construct the finite equations of a group when those of its infinitesimal transformation are known. The simultaneous system to be integrated in this case has the form

$$\frac{d\bar{x}_1}{0}=\frac{d\bar{y}_1}{1}=dt,$$

with the initial values $\bar{x}$, $\bar{y}$, 0. The integration of this simple system gives the finite equation in the form

$$\bar{x}_1=\bar{x},\quad \bar{y}_1=\bar{y}+t;$$

that is, the group is a group of translations parallel to the $\bar{y}$-axis.

6. The theorem of the preceding paragraph can be proved by using the finite equations of the group. We have found in a previous article of this series that the finite equations

$$x_1=\varphi(x,\,y,\,t),\quad y_1=\psi(x,\,y,\,t) \tag{24}$$

of a one parameter group are found by integrating a certain simultaneous system in the form*

$$\Omega(x_1,\,y_1)=\Omega(x,\,y),$$

$$W(x_1,\,y_1)+t=W(x,\,y).$$

*The reader will observe that this is the same theorem made use of in the paragraph immediately preceding.

Now if the functions Ω and W be introduced as new variables in place of x and y, and put

$$\bar{x}=\Omega(x, y), \quad \bar{y}=W(x, y),$$

$$\bar{x}_1=\Omega(x_1, y_1), \quad \bar{y}_1=W(x_1, y_1),$$

the group takes the simple form

$$\bar{x}_1=\bar{x}, \quad \bar{y}_1=\bar{y}+t; \tag{25}$$

that is, the form of a group of translations.

The new variables $\bar{x}$ and $\bar{y}$ are called the *canonical variables* of the one parameter group (24), and (25) is called the *canonical form* of (24).

The reader will find many interesting details relative to the points and theorems here discussed, in the third chapter of LIE's lectures on differential equations.

Princeton University, 21 February, 1898.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

84. Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.

Show how to find sides, integral, fractional, and irrational for twenty-four triangles, each one containing 330 square yards.

Solution by the PROPOSER.

1. $\triangle^2=330^2=2\times2\times3\times3\times5\times5\times11\times11=(5\times11)(2\times3\times5)(2\times11)\times3=55\times30.22\times3$. ∴ Sides of triangle are 25, 33, and 52.

2. $\triangle^2=330^2=(2\times5\times11)(3\times3\times11)(2\times5)=110\times99\times10\times1$. ∴ Sides of $\triangle$ are 11, 100, and 109.

3. $\triangle^2=330^2=(2\times5\times5)(3\times11)(11)(2\times3)=50\times33\times11\times6$. ∴ Sides of $\triangle$ are 17, 39, 44.

4. $330^2=(2\times3\times11)(5\times11)(2\times3)(5)=66\times55\times6\times5$. ∴ Sides of $\triangle$ are 11, 60, and 61.

5. $330^2=(2\times2\times2\times2\times3)\left(\frac{5\times11}{2}\right)(3\times5)(\tfrac{11}{2})=48\times27\tfrac{1}{2}\times15\times5\tfrac{1}{2}$. ∴ Sides of $\triangle$ are $20\tfrac{1}{2}$, 33, and $42\tfrac{1}{2}$.

6. $330^2=(2\times2\times11)(2\times2\times5)\left(\frac{3\times11}{2}\right)\left(\frac{3\times5}{2}\right)=44\times20\times16\frac{1}{2}\times7\frac{1}{2}$. $\therefore$ Sides of $\triangle$ are 24, $27\frac{1}{2}$, and $36\frac{1}{2}$.

7. $330^2=(2\times2\times3\times5)\left(\frac{3\times5\times11}{2\times2}\right)(2\times2\times2\times2)\left(\frac{11}{2\times2}\right)=60\times41\frac{1}{4}\times16\times2\frac{3}{4}$. $\therefore$ Sides of $\triangle$ are $18\frac{3}{4}$, 44, and $57\frac{1}{4}$.

8. $330^2=(3\times3\times11)(2\times3\times3\times5)\left(\frac{2\times11}{3}\right)(\frac{5}{3})=99\times90\times7\frac{1}{3}\times1\frac{2}{3}$. $\therefore$ Sides of $\triangle$ are 9, $91\frac{2}{3}$, and $97\frac{1}{3}$.

9. $\triangle^2=165^2=3\times3\times5\times5\times11\times11=(3\times11)(2\times2\times5)\left(\frac{3\times5}{2}\right)(\frac{11}{2})=33\times20\times7\frac{1}{2}\times5\frac{1}{2}$. $\therefore$ Sides of $\triangle=165$ are 13, $25\frac{1}{2}$, and $27\frac{1}{2}$, and sides of $\triangle=330$ are $13\sqrt{2}$, $25\frac{1}{2}\sqrt{2}$, and $27\frac{1}{2}\sqrt{2}$.

10. $165^2=(2\times3\times5)\left(\frac{5\times11}{2\times2}\right)\left(\frac{3\times11}{2\times2}\right)(2\times2\times2)=30\times13\frac{3}{4}\times8\frac{1}{4}\times8$. $\therefore$ Sides of $\triangle$ are $16\frac{1}{4}$, $21\frac{3}{4}$, and 22. Sides of similar $\triangle$ containing 330 must be $16\frac{1}{4}\sqrt{2}$, $21\frac{3}{4}\sqrt{2}$, and $22\sqrt{2}$.

11. $\triangle^2=110^2=2\times2\times5\times5\times11\times11=(2\times2\times2\times3)\left(\frac{5\times11}{2\times3}\right)\left(\frac{3\times5}{2}\right)\left(\frac{2\times11}{3}\right)$ $=24\times9\frac{1}{6}\times7\frac{1}{2}\times7\frac{1}{3}$. $\therefore$ Sides of $\triangle=110$ are $14\frac{5}{6}$, $16\frac{1}{2}$, and $16\frac{2}{3}$. Sides of $\triangle=330$ are $14\frac{5}{6}\sqrt{3}$, $16\frac{1}{2}\sqrt{3}$, and $16\frac{2}{3}\sqrt{3}$.

12. $110^2=\left(\frac{2\times5\times11}{3}\right)(2\times3\times5)(\frac{11}{3})(3)=36\frac{2}{3}\times30\times3\frac{2}{3}\times3$. $\therefore$ Sides are $6\frac{2}{3}$, 33, and $33\frac{2}{3}$, and sides of similar $\triangle=330$ must be $6\frac{2}{3}\sqrt{3}$, $33\sqrt{3}$, and $33\frac{2}{3}\sqrt{3}$.

13. $\triangle^2=66^2=2\times2\times3\times3\times11\times11=(2\times11)(11)(3\times3)(2)=22\times11\times9\times2$. $\therefore$ Sides of $\triangle=66$ are 11, 13, and 20, and sides of $\triangle=330$ are $11\sqrt{5}$, $13\sqrt{5}$, and $20\sqrt{5}$.

14. $66^2=(2\times3\times3\times3)(3\times3\times3)\left(\frac{2\times11\times11}{3\times3}\right)\left(\frac{1}{3\times3}\right)=54\times27\times26\frac{8}{9}\times\frac{1}{9}$. $\therefore$ Sides are 27, $27\frac{1}{9}$, and $53\frac{8}{9}$, and sides of $\triangle=330$ are $27\sqrt{5}$, $27\frac{1}{9}\sqrt{5}$, and $53\frac{8}{9}\sqrt{5}$.

15. $66^2=(3\times11)\left(\frac{3\times11}{2}\right)(2\times2\times2\times2)(\frac{1}{2})=33\times16\frac{1}{2}\times16\times\frac{1}{2}$. $\therefore$ Sides of $\triangle=66$ are $16\frac{1}{2}$, 17, and $32\frac{1}{2}$. Necessarily, sides of similar $\triangle=330$ are $16\frac{1}{2}\sqrt{5}$, $17\sqrt{5}$, and $32\frac{1}{2}\sqrt{5}$.

16. $66^2=(2\times2\times2\times3)\left(\frac{3\times11}{2}\right)(\frac{11}{2})(2)=24\times16\frac{1}{2}\times5\frac{1}{2}\times2$. $\therefore$ Sides are $7\frac{1}{2}$, $18\frac{1}{2}$, and 22, and sides of similar $\triangle=330$ are $7\frac{1}{2}\sqrt{5}$, $18\frac{1}{2}\sqrt{5}$, $22\sqrt{5}$.

17. $\triangle^2=55^2=(5\times5\times11\times11)=(2\times3\times3)\left(\frac{5\times11}{2\times3}\right)(\frac{11}{2})\left(\frac{2\times5}{3}\right)=18\times9\frac{1}{6}\times5\frac{1}{2}\times3\frac{1}{3}$. $\therefore$ Sides are $8\frac{5}{6}$, $12\frac{1}{2}$, and $14\frac{2}{3}$, and sides of $\triangle=330$ are $8\frac{5}{6}\sqrt{6}$, $12\frac{1}{2}\sqrt{6}$, and $14\frac{2}{3}\sqrt{6}$.

18. $55^2=(2\times2\times5)\left(\frac{5\times11}{2\times3}\right)(3\times3)\left(\frac{11}{2\times3}\right)=20\times9\frac{1}{6}\times9\times1\frac{5}{6}$. $\therefore$ Sides of $\triangle=55$ are $10\frac{5}{6}$, 11, and $18\frac{1}{6}$, and sides of $\triangle=330$ are $10\frac{5}{6}\sqrt{6}$, $11\sqrt{6}$, $18\frac{1}{6}\sqrt{6}$.

19. $\triangle^2=(47\tfrac{1}{7})^2=(\tfrac{330}{7})^2=(3\times 11)(2\times 3\times 5)(\tfrac{11}{7})\left(\frac{2\times 5}{7}\right)=33\times 30\times 1\tfrac{4}{7}\times 1\tfrac{3}{7}$.
$\therefore$ Sides are 3, $31\tfrac{3}{7}$, and $31\tfrac{4}{7}$, and sides of $\triangle=330$ are $3\sqrt{7}$, $31\tfrac{3}{7}\sqrt{7}$, and $31\tfrac{4}{7}\sqrt{7}$.

20. $\triangle^2=(47\tfrac{1}{7})^2=(\tfrac{330}{7})=\left(\frac{2\times 5\times 11}{7}\right)\left(\frac{3\times 11}{7}\right)(2\times 3)(5)=15\tfrac{5}{7}\times 4\tfrac{5}{7}\times 6\times 5$.
$\therefore$ Sides are 11, $9\tfrac{5}{7}$, and $10\tfrac{5}{7}$, and sides of $\triangle=330$ are $11\sqrt{7}$, $9\tfrac{5}{7}\sqrt{7}$, and $10\tfrac{5}{7}\sqrt{7}$.

21. $(47\tfrac{1}{7})^2=(\tfrac{330}{7})^2=(2\times 11)\left(\frac{2\times 2\times 2\times 3\times 5}{7}\right)(\tfrac{5}{2})\left(\frac{3\times 11}{2\times 7}\right)=22\times 17\tfrac{1}{7}\times 2\tfrac{1}{2}\times 2\tfrac{5}{14}$. $\therefore$ Sides are $4\tfrac{6}{7}$, $19\tfrac{1}{2}$, and $19\tfrac{9}{14}$, and sides of $\triangle=330$ are $4\tfrac{6}{7}\sqrt{7}$, $19\tfrac{1}{2}\sqrt{7}$, $19\tfrac{9}{14}\sqrt{7}$.

22. $(47\tfrac{1}{7})^2=(\tfrac{330}{7})^2=\left(\frac{3\times 11}{2}\right)\left(\frac{2\times 2\times 3\times 5}{7}\right)(2\times 2)\left(\frac{5\times 11}{2\times 7}\right)=16\tfrac{1}{2}\times 8\tfrac{4}{7}\times 4\times 3\tfrac{13}{14}$. $\therefore$ Sides are $7\tfrac{13}{14}$. $12\tfrac{1}{2}$, and $12\tfrac{4}{7}$, and sides of $\triangle=330$ are $7\tfrac{13}{14}\sqrt{7}$, $12\tfrac{1}{2}\sqrt{7}$, and $12\tfrac{4}{7}\sqrt{7}$.

23. $\triangle^2=33^2=(3\times 3\times 11\times 11)=\left(\frac{3\times 11}{2}\right)(11)(2\times 2)(\tfrac{3}{2})=16\tfrac{1}{2}\times 11\times 4\times 1\tfrac{1}{2}$.
$\therefore$ Sides are $5\tfrac{1}{2}$, $12\tfrac{1}{2}$, and 15, and sides of similar $\triangle=330$ are $5\tfrac{1}{2}\sqrt{10}$, $12\tfrac{1}{2}\sqrt{10}$, and $15\sqrt{10}$.

24. $33^2=\left(\frac{3\times 3\times 5}{2}\right)\left(\frac{11\times 11}{2\times 5}\right)(2\times 5)(\tfrac{2}{5})=22\tfrac{1}{2}\times 12\tfrac{1}{10}\times 10\times \tfrac{2}{5}$. $\therefore$ Sides are $10\tfrac{2}{5}$, $12\tfrac{1}{2}$, and $22\tfrac{1}{10}$, and sides of $\triangle=330$ are $10\tfrac{2}{5}\sqrt{10}$, $12\tfrac{1}{2}\sqrt{10}$, and $22\tfrac{1}{10}\sqrt{10}$.

25. $\triangle^2=30^2=2\times 2\times 3\times 3\times 5\times 5=(3\times 5)(2\times 5)(3)(2)=15\times 10\times 3\times 2$.
$\therefore$ Sides of $\triangle=30$ are 5, 12, and 13, and sides of $\triangle=330$ are $5\sqrt{11}$, $12\sqrt{11}$, and $13\sqrt{11}$.

26. $30^2=(2\times 2\times 5)(2\times 2\times 3)\left(\frac{3\times 5}{2}\right)(\tfrac{1}{2})=20\times 12\times 7\tfrac{1}{2}\times \tfrac{1}{2}$. $\therefore$ Sides are 8, $12\tfrac{1}{2}$, and $19\tfrac{1}{2}$, and sides of $\triangle=330$ are $8\sqrt{11}$, $12\tfrac{1}{2}\sqrt{11}$, and $19\tfrac{1}{2}\sqrt{11}$.

27. $30^2=(2\times 2\times 2\times 2\times 2)(2\times 3\times 5)\left(\frac{5}{2\times 2}\right)\left(\frac{3}{2\times 2}\right)=32\times 30\times 1\tfrac{1}{4}\times \tfrac{3}{4}$.
$\therefore$ Sides of $\triangle=30$ are 2, $30\tfrac{3}{4}$, and $31\tfrac{1}{4}$, and sides of $\triangle=330$ are $2\sqrt{11}$, $30\tfrac{3}{4}\sqrt{11}$, and $31\tfrac{1}{4}\sqrt{11}$.

28. $\triangle^2=(25\tfrac{5}{13})^2=(\tfrac{330}{13})^2=(3\times 5)(11)\left(\frac{2\times 3\times 5}{13}\right)\left(\frac{2\times 11}{13}\right)=15\times 11\times 2\tfrac{4}{13}\times 1\tfrac{9}{13}$. $\therefore$ Sides are 4, $12\tfrac{9}{13}$, and $13\tfrac{4}{13}$, and sides of $\triangle=330$ are $4\sqrt{13}$, $12\tfrac{9}{13}\sqrt{13}$, and $13\tfrac{4}{13}\sqrt{13}$.

29. $(\tfrac{330}{13})^2=\left(\frac{2\times 3\times 5\times 11}{13}\right)(2\times 11)(3)(\tfrac{5}{13})=25\tfrac{5}{13}\times 22\times 3\times \tfrac{5}{13}$. $\therefore$ Sides are $3\tfrac{5}{13}$, $22\tfrac{5}{13}$, and 25, and sides of $\triangle=330$ are $3\tfrac{5}{13}\sqrt{13}$, $22\tfrac{5}{13}\sqrt{13}$, and $25\sqrt{13}$.

30. $\triangle^2=(23\tfrac{4}{7})^2=(\tfrac{165}{7})^2=\frac{3\times 3\times 5\times 5\times 11\times 11}{7\times 7}=\left(\frac{3\times 5\times 11}{2\times 7}\right)(\tfrac{11}{2})\left(\frac{2\times 2\times 3\times 5}{2\times 7}\right)(2)=11\tfrac{11}{14}\times 5\tfrac{1}{2}\times 4\tfrac{4}{14}\times 2$. $\therefore$ Sides of $\triangle=23\tfrac{4}{7}$ are $6\tfrac{2}{7}$, $7\tfrac{1}{2}$, and $9\tfrac{11}{14}$, and sides of $\triangle=330$ are $6\tfrac{2}{7}\sqrt{14}$, $7\tfrac{1}{2}\sqrt{14}$, and $9\tfrac{11}{14}\sqrt{14}$.

31. $(23\frac{4}{7})^2=(\frac{165}{7})^2=(2\times11)\left(\frac{3\times5\times11}{2\times7}\right)(2\times5)\left(\frac{3}{2\times7}\right)=22\times\frac{11}{14}\times10\times\frac{3}{14}$.

$\therefore$ Sides are $10\frac{3}{14}$, 12. and $21\frac{11}{14}$, and sides of $\triangle=330$ are $10\frac{3}{14}\sqrt{14}$, $12\sqrt{14}$, and $21\frac{11}{14}\sqrt{14}$.

32. $\triangle^2=22^2=2\times2\times11\times11=(2\times11)(2\times3\times3)(\frac{11}{3})(\frac{1}{3})=22\times18\times3\frac{2}{3}\times\frac{1}{3}$. $\therefore$ Sides are 4, $18\frac{1}{3}$. and $21\frac{2}{3}$, and sides of $\triangle=330$ are $4\sqrt{15}$, $18\frac{1}{3}\sqrt{15}$, and $21\frac{2}{3}\sqrt{15}$.

33. $\triangle^2=15^2=3\times3\times5\times5=(2\times5)(2\times3)(\frac{5}{2})(\frac{3}{2})=10\times6\times2\frac{1}{2}\times1\frac{1}{2}$. $\therefore$ Sides are 4, $7\frac{1}{2}$, and $8\frac{1}{2}$, and, of course, sides of $\triangle=330$ are $4\sqrt{22}$, $7\frac{1}{2}\sqrt{22}$, and $8\frac{1}{2}\sqrt{22}$.

34. $\triangle^2=10^2=2\times2\times5\times5=(3\times3)(5)\left(\frac{2\times5}{3}\right)(\frac{2}{3})=9\times5\times3\frac{1}{3}\times\frac{2}{3}$. $\therefore$ Sides are 4, $5\frac{2}{3}$, and $8\frac{1}{3}$, and sides of $\triangle=330$ are $4\sqrt{33}$, $5\frac{2}{3}\sqrt{33}$, and $8\frac{1}{3}\sqrt{33}$.

ALGEBRA.

79. **Proposed by C. W. M. BLACK, A. M., Professor of Mathematics, Wesleyan Academy, Wilbraham, Mass.**

Of n persons A, B, C, etc., A gives to the others as much as each of them already has; then B gives to the others as much as each then has; and so on for each in turn. Finally, A, B, C, etc., have respectively a, b, c, etc., dollars. How much had each at first?

I. Solution by the PROPOSER.

Let $a+b+c+$ etc......$=s$, which is constant. Consider the rth person, K, who has at the end k dollars. Let x=number of dollars he had at first. Before he gives away any, his original amount is doubled $r-1$ times and comes to equal $2^{r-1}x$. At that time all the rest must together have $s-2^{r-1}x$, and if he gives away that amount he has left $2^{r-1}x-(s-2^{r-1}x)=2^rx-s$. Afterwards his money is doubled $n-r$ times and finally equals $2^{n-r}(2^rx-s)=k$.

$\therefore x=(k+2^{n-r}s)/2^n$.

II. Solution by I. H. BRYANT, A. M., Instructor in Mathematics in Fort Smith High School, Fort Smith, Ark.; G. B. M. ZERR, A. M., Ph. D., Russell College, Lebanon, Va.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let x_1, x_2, x_3, x_4, ..., x_n be the respective amounts each had at first.

Let $s=x_1+x_2+x_3+\ldots\ldots+x_n=a+b+c+\ldots\ldots$ (1).

$\therefore 2^{n-1}(2x_1-s)=a$ .....(2).

$2^{n-2}(4x_2-s)=b$ ......(3).

$2^{n-3}(8x_3-s)=c$ ......(4).

....

$(2^nx_n-s=n$.

(2), (3), (4), (5), etc., in (1) gives, $x_1=(a+2^{n-1}s)/2^n$, $x_2=(b+2^{n-2}s)/2^n$, $x_3=(c+2^{n-3}s)/2^n$, $x_4=(d+2^{n-4}s)/2^n$, ...$x_n=(n+s)/2^n$.

III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let x_1, x_2, x_3, ... x_{n-1}, x_n=the money that A, B, C, etc., respectively, had at first. Put $x_n=r$, $x_{n-1}=2r-s$, $x_{n-2}=2x_{n-1}-s=4r-3s$, $x_{n-3}=2x_{n-2}-s=8r-7s$, etc., etc. The law that governs the forming of these values makes $x_1=2^{n-1}(r-s)+s$; $x_2=2^{n-2}(r-s)+s$; $x_3=2^{n-3}(r-s)+s$; $x_4=2^{n-4}(r-s)+s$;

etc., etc. Whence we deduce, $a=2^{n-1}[r-(n-1)s]$; $b=2^{n-2}\{r-[n-(2^2-1)]s\}$; $c=2^{n-3}\{r-[n-(2^3-1)]s\}$; $d=2^{n-4}\{r-[n-(2^4-1)]s\}$; etc., etc. But in order that all values are positive, we have the condition $r>(n-1)s$. Therefore, put $r=(n-1)s+t$, and we obtain,

$a=2^{n-1}t$, $\quad x_1=2^{n-1}[s(n-2)+t]+s$,

$b=2^{n-2}(2s+t)$, $\quad x_2=2^{n-2}[s(n-2)+t]+s$,

$c=2^{n-3}2(2^2-1)s+t$, $\quad x_3=2^{n-3}[s(n-2)+t]+s$,

$d=2^{n-4}2(2^3-1)s+t$, etc. $\quad x_4=2^{n-4}[s(n-2)+t]+s$, etc.

s and t may be any values, integral or fractional ; but s may also be *zero*. When $s=0$, the money each had at first equals, respectively, what he had at the end of the transaction ; or $a=x_1=2^{n-1}t$; $b=x_2=2^{n-1}t$; ...... $x_{n-1}=2^2t$; $x_n=2t$. When $t=2s$, all have an equal sum at the end of the transaction ; or $2^ns=a=b=c$, etc. For obtaining integral values this form of solution is preferable.

IV. **Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, N. J.**

This problem is concerned with a double n-fold series. Let A, B, C, etc., represent the amounts which each person has to start with ; then after the several exchanges, we have : A's nth term$=2^{n-1}[2A-\Sigma(A)]=a$; where $\Sigma(A)$ means $(A+B+C+\ldots\ldots$ to n letters) ; B's nth term equals $2^{n-2}[4B-\Sigma(A)]=b$; C's nth term$=2^{n-3}[8C-\Sigma(A)]=c$; and so on for R's nth term$=2^{n-r}[2^rR-\Sigma(A)]=\rho$; and N's nth term$=2^{n-n}[2^nN-\Sigma(A)]=\nu$.

Expanding, we may write,

$$\left.\begin{array}{r}
(2^n-2^{n-1})A-2^{n-1}B-2^{n-1}C\ldots\ldots-2^{n-1}R\ldots\ldots-2^{n-1}N=a \\
-2^{n-2}A+(2^n-2^{n-2})B-2^{n-2}C\ldots\ldots-2^{n-2}R\ldots\ldots-2^{n-2}N=b \\
-2^{n-3}A-2^{n-3}B+(2^n-2^{n-3})C\ldots\ldots-2^{n-3}R\ldots\ldots-2^{n-3}N=c \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
-2^{n-r}A-2^{n-r}B-2^{n-r}C\ldots\ldots+(2^n-2^{n-r})R\ldots\ldots-2^{n-r}N=\rho \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
-2^{n-n}A-2^{n-n}B-2^{n-n}C\ldots\ldots-2^{n-n}R\ldots\ldots+(2^n-2^{n-n})N=\nu
\end{array}\right\}\text{I.}$$

Whence by determinants,

$$\Delta.A=\left|\begin{array}{lllll}
a, & -2^{n-1}, & -2^{n-1}, & \ldots.-2^{n-1}, & \ldots.-2^{n-1} \\
b, & 2^n-2^{n-2}, & -2^{n-2}, & \ldots.-2^{n-2}, & \ldots.-2^{n-2} \\
c, & -2^{n-3}, & 2^n-2^{n-3}, & \ldots.-2^{n-3}, & \ldots.-2^{n-3} \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
\rho, & -2^{n-r}, & -2^{n-r}, & \ldots.2^n-2^{n-r}, & \ldots.-2^{n-r} \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
\nu, & -1, & -1, & \ldots.-1, & \ldots.2^n-1
\end{array}\right|$$

Where Δ is the determinant of the coefficients.

Also solved by *H. C. WILKES*.

80. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

Solve $1+x^4=a(1+x)^4$.

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; P. S. BERG, A. M., Principal of Schools, Larimore, N. D.; J. OWEN MAHONEY, B. E., M. Sc., Graduate Fellow in Mathematics in Vanderbilt University, Lynnville, Tenn.; CHAS. A. JONES, Torrance, Miss.; and the PROPOSER.

$1+x^4=a+4ax+6ax^2+4ax^3+ax^4$.

$x^2+(1/x^2)=(a/x^2)+(4a/x)+6a+4ax+ax^2$.

$[x+(1/x)]^2=a[x+(1/x)]^2+4a[x+(1/x)]+4a+2\ldots\ldots(1)$.

Let $x+(1/x)=y\ldots\ldots(2)$; $\therefore\ x^2-xy=1$.

$\therefore\ x=\frac{1}{2}[y\pm\sqrt{(y^2-4)}]\ldots\ldots(3)$. (2) in (1) gives

$y^2+[4a/(a-1)]y=-[(4a+2)/(a-1)]$.

$\therefore\ y=[\pm\sqrt{\ }(2a+2)-2a]/(a-1)\ldots\ldots(4)$. (4) in (3) gives

$x=[1/2(a-1)]\{-2a\pm\sqrt{[2a+2]}\pm\sqrt{[10a-2\pm4a\sqrt{(2a+2)}]}\}$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.; and F. M. McGAW, A. M., Professor of Mathemetics in Bordentown Military Institute, Bordentown, N. J.

Expanding and arranging we get, dividing by $a-1$,

$$x^4+\frac{4a}{a-1}x^3+\frac{6a}{a-1}x^2+\frac{4a}{a-1}x+1=0,$$

a reciprocal equation which is easily reduced by dividing by x^2, thus

$$[x^2+(1/x^2)]+\frac{4a}{a-1}[x+(1/x)]+\frac{6a}{a-1}=0,$$

Putting $x+(1/x)=y$, we shall have

$$y^2+\frac{4a}{a-1}y=-\frac{4a+2}{a-1},\ \text{whence}\ y=-\frac{2a}{a-1}\pm\frac{1}{a-1}\sqrt{2(a+1)},$$

and then

$$x=-\frac{a}{a-1}\pm\frac{1}{2(a-1)}\sqrt{2(a+1)}\pm\frac{1}{2(a-1)}\sqrt{2(2a^2+a+1)\mp4a\sqrt{2(a+1)}}.$$

Also solved by *A. H. BELL.*

GEOMETRY.

84. Proposed by FREDERICK R. HONEY, Ph. B., Instructor in Mathematics in Trinity College, New Haven, Conn.

Find the locus of a point which will trisect all arcs having a common chord.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

In what follows the arc of the circle is assumed.

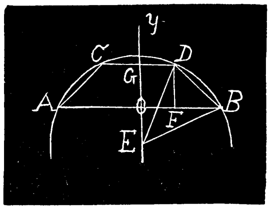
Let O be the origin, $EO=z$, $AB=2c$, $\overline{AC}=\overline{CD}=\overline{DB}=n$.

Then $BF=\frac{1}{2}(2c-n)$, $OF=\frac{1}{2}n$, $DF=\sqrt{n^2-\frac{1}{4}(2c-n)^2}$, $EB^2=c^2+z^2$, $ED^2=EB^2=c^2+z^2$, $EG^2=c^2+z^2-\frac{1}{4}n^2$.

$EG=$ also $EO+DF=z+\frac{1}{2}\sqrt{4n^2-(2c-n)^2}$.

$\therefore\ c^2+z^2-\frac{1}{4}n^2=z^2+z\sqrt{4n^2-(2c-n)^2}+n^2-c^2+cn-\frac{1}{4}n^2$,

or $2c^2-(n^2+cn)=z\sqrt{3n^2-4c^2+4cn}$ ....(1).



Equation to circle is $x^2+(y+z)^2=c^2+z^2$,

or $x^2+y^2+2yz=c^2$. $\therefore\ z=(c^2-x^2-y^2)/2y$ .....(2).

$OF^2+FD^2=x^2+y^2=\frac{1}{4}n^2+n^2-\frac{1}{4}(2c-n)^2$.

$\therefore\ n^2+cn=x^2+y^2+c^2$.

$\therefore\ n=\frac{1}{2}(\sqrt{4x^2+4y^2+5c^2}-c)$ ....(3).

(2) and (3) in (1) gives

$$2c^2-x^2-y^2-c^2=\frac{c^2-x^2-y^2}{2y}\sqrt{4x^2+4y^2-\tfrac{1}{4}(\sqrt{4x^2+4y^2+5c^2}-c)^2}.$$

$$\therefore\ 4y^2=3x^2+3y^2-\tfrac{3}{2}c^2+\tfrac{1}{2}c\sqrt{4x^2+4y^2+5c^2},$$

$$(2y^2-6x^2+3c^2)^2=c^2(4x^2+4y^2+5c^2).$$

$$(y^2-3x^2)^2+2c^2(y^2-5x^2)+c^4=0.$$

II. Solution by the PROPOSER.

Let *abcd*—Fig. 1—be any arc, and *ad* the chord. Bisect *ad* at *o*, and draw *eof* perpendicular to it. We wish to find the locus of a point *c* whose distance from a given straight line *ef* is one half the distance from a given point *d*.

In order to write the equation of this curve, refer it to the coördinate axes *ad* (axis of X) and *ef* (axis of Y), intersecting at the origin *o*.

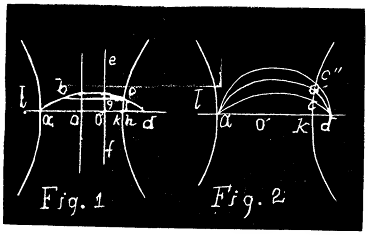


Fig. 1 Fig. 2

Let $gc=x$. Therefore, from the definition $cd=2x$. Let $od=D$. $\therefore\ hd=D-x$.

Let $ch=y$. $\therefore\ (2x)^2=y^2+(D-x)^2$ or $4x^2=y^2+D^2-2Dx+x^2$.

$\therefore\ y^2-3x^2+D^2-2Dx=0$....(I). This is the equation of an hyperbola whose center is on the axis of abscissas. In order to determine the position of the center, eliminate the x term, and find the distance from the origin o to a new origin o'. Let E=distance from o to o'. $\therefore\ x=x'+E$. Substituting this value of x in equation (I), $y^2-3(x'+E)^2+D^2-2D(x'+E)=0$; or $y^2-3x'^2-6Ex'-3E^2+D^2-2Dx'-2DE=0$......(II). In this equation the x' terms should disappear.

$\therefore\ 6Ex'-2Dx'=0$. $\therefore\ E=-\frac{1}{3}D$.

That is, the distance from the origin o to the new origin, or the center of the hyperbola, o', is equal to one third of the distance from o to d; and the minus sign indicates that the measurement should be laid off to the left of the origin o. Substituting this value of E in equation II, and omitting accents, we have $y^2-3x^2+2x-\frac{1}{3}D^2+D^2-2x+\frac{2}{3}(D^2)=0$.

∴ $y^2-3x^2=-\frac{1}{3}(4D^2)$. This is the equation of an hyperbola referred to its center o' as the origin. To write it in the ordinary form, that is, in terms of the transverse and conjugate axes, multiply each term by C; *i. e.* let $\sqrt{C}$=semi-transverse axis. Thus $Cy^2-3Cx^2=-\frac{1}{3}(4CD^2)$.

When in this form the product of the coefficients of the x^2 and y^2 terms should be equal to the remaining term. That is $-3C^2=-\frac{1}{3}(4CD^2)$......(III). ∴ $C=\frac{1}{9}(4D^2)$, and equation III becomes $\frac{1}{9}(4D^2)y^2-\frac{1}{3}(4D^2)x^2=-\frac{1}{27}(16D^4)$.

The semi-transverse axis $=\sqrt{[\frac{1}{9}(4D^2)]}=\frac{1}{3}(2D)$. The semi-conjugate axis $=\sqrt{[\frac{1}{3}(4D^2)]}=2D/\sqrt{3}$.

Since the distance from the center of the curve to either focus is equal to the square root of the sum of the squares of the semi-axes, the distance from o' to either focus $=\sqrt{\{[\frac{1}{9}(4D^2)]+[\frac{1}{3}(5D^2)]\}}=\frac{1}{3}(4D)$. We can therefore make the following construction—Fig. 2. Draw *ad* the chord of the arc *acd*. Trisect *ad* at o' and k. Produce *da* to l, making $al=ao'=o'k=kd$. With *ak* as a transverse axis, and l and d as foci, construct the branch of the hyperbola $kcc'c''$, which will intersect all arcs having the common chord *ad* at c, c', c'', etc., making the arcs cd, $c'd$, $c''d$, respectively, equal to one-third of the arcs acd, $ac'd$, $ac''d$, etc.

CALCULUS.

67. Proposed by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, O.

A man starts to walk at a uniform rate across a draw-bridge just as it begins to move. He walks the full length of the bridge and back, in the same time that it takes the bridge to make a half revolution. How for does he ride, the length of the bridge being 250 feet, and its velocity uniform about a center axis?

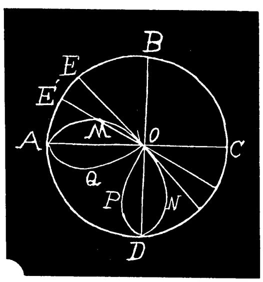
I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

In my solution of problem 45, I deduced the equation, $\rho=r(m-\theta)/m$. Let $m=\frac{1}{4}\pi$. ∴ $\rho=(r/\pi)(\pi-4\theta)$ is the equation to the man's path in space.

$$\therefore\ S=\frac{2r}{\pi}\int_0^{\frac{1}{2}\pi}\sqrt{16+(\pi-4\theta)^2}\,d\theta=\tfrac{1}{2}r\sqrt{16+\pi^2}+\frac{8r}{\pi}\left(\frac{\pi+\sqrt{(16+\pi^2)}}{4}\right)$$

$$=\tfrac{125}{2}\sqrt{16+\pi^2}+\frac{1000}{\pi}\log\frac{\pi+\sqrt{(16+\pi^2)}}{4}.\quad \therefore\ S=547.468 \text{ feet.}$$

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.



According to the conditions of the problem, A being the starting point of the man, he will be at O, going in the curve AMO, when the draw-bridge has turned through $\angle AOE=45°$; when the latter has turned through $\angle AOB=90°$, the man is in D, having passed through the curve OND; after the bridge has turned through $\angle AOF=135°$, the man is at O again, having moved through the curve DPO, and after the

draw-bridge has made a semi-revolution, he is back at A, having been swept through the curve AQO. Choosing O for the origin of polar coödinates, we have for the curve to the polar equation $r=4R\theta/\pi$, R being the radius of the revolving draw-bridge. The required path of the man is

$$=4\int_0^{\frac{1}{4}\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta=\frac{16R}{\pi}\int_0^{\frac{1}{4}\pi}\sqrt{\theta^2+1}\,d\theta=\frac{8R}{\pi}[\theta\sqrt{\theta^2+1}+$$

$$\log(\theta+\sqrt{\theta^2+1})\Big]_0^{\frac{1}{4}\pi}=\tfrac{1}{2}R\left[\sqrt{\pi^2+16}+\frac{16}{\pi}\log\left(\frac{\pi+\sqrt{(\pi^2+16)}}{4}\right)\right];$$

for $R=125$ feet, we get this length$=547.45$ feet.

III. **Solution by J. M. BANDY, A. M., Civil Engineer for the Roads and Bridges of Guilford County, Greensboro, N. C.**

Let M be any position of the man, let (ρ, θ) denote the polar coördinates of M, and let r=radius of the bridge. When A shall have revolved through 45° the man will be at O, the center; and the limits of θ for $\frac{1}{4}$ the curve are 0 and $\frac{1}{4}\pi$.

Since the man and the motion of the bridge are uniform, AE' and $E'M$ are in a constant ratio, which denote by n. But $AE'=r6$, $E'M=r-\rho$, and $n=\frac{1}{4}\pi$. Hence, $r[1-(4/\pi)\theta]$......(1). By the theory of curves,

$$S=\int\left(\rho^2+\frac{d\rho^2}{d\theta^2}\right)^{\frac{1}{2}}d\theta.\text{ From (1), }\rho^2=r^2[1-(4/\pi)\theta]^2,\text{ and }\frac{d\rho^2}{d\theta^2}=\frac{4r^2}{\pi^2}.$$

Substituting in formula,

$$S=4\int_0^{\frac{1}{4}\pi}\left[\frac{4r^2}{\pi^2}+\frac{r^2}{\pi^2}(\pi-4\theta)^2\right]^{\frac{1}{2}}d\theta,=\frac{4r}{\pi}\int_0^{\frac{1}{4}\pi}[4+(\pi-4\theta)^2]^{\frac{1}{2}}\,d\theta,$$

$$=\frac{4r}{\pi}\left[\frac{(\pi-4\theta)}{2}\sqrt{4+(\pi-4\theta)^2}+\tfrac{1}{2}\log\left((\pi-4\theta)+\sqrt{4+(\pi-4\theta)^2}\right)\right]_0^{\frac{1}{4}\pi},$$

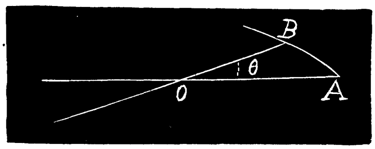
$$=\tfrac{1}{2}r\sqrt{16+\pi^2}+(4r/\pi)\log\left(\frac{\pi+\sqrt{(16+\pi^2)}}{4}\right)$$

$$=\tfrac{125}{2}\sqrt{16+\pi^2}+\frac{1000}{\pi}\log\left(\frac{\pi+\sqrt{(16+\pi^2)}}{4}\right).$$

$S=547.45$ feet.

IV. **Solution by C. W. M. BLACK, A. M., Professor of Mathematics, Wesleyan Academy, Wilbraham, Mass.**

Let a=number of complete revolutions made while he walks the full length. $b=2OA$=length of draw-bridge. Let AB be curve traversed. $OB=\rho$, $\angle AOB=\theta$.



Then $\frac{1}{2}b-\rho : b=\theta : 2a\pi$.

$$\rho=b\left(\tfrac{1}{2}-\frac{\theta}{2a\pi}\right).\quad\frac{d\rho}{d\theta}=-\frac{b}{2a\pi}.$$

s (distrance traversed in passing to end of bridge)

$$=\int_0^{2a\pi}\left[b^2\left(\tfrac{1}{2}-\frac{\theta}{2a\pi}\right)^2+\frac{b^2}{4a^2\pi^2}\right]^{\frac{1}{2}}d\theta=(\tfrac{1}{2}b)\sqrt{1+a^2\pi^2}-$$

$$\frac{b}{2a\pi}\log_e(\sqrt{1+a^2\pi^2}-a\pi).\quad b=250.\quad a=\tfrac{1}{4}.$$

$\therefore\ 2s=250[1+(\tfrac{1}{16}\pi^2)]-(1000/\pi)\log_e\{\sqrt{[1+(\tfrac{1}{16}\pi^2)]}-\tfrac{1}{4}\pi\}=547.6+$ feet.

[See also solutions of problems 41, 45, and 50, published in previous numbers of MONTHLY. EDITOR.]

MECHANICS.

56. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics in Manual Training School, Philadelphia, Pa.

"Hey-diddle-diddle, the cat and the fiddle,
The cow jumped over the moon."

Taking the weight of the cow to be 600 pounds, the initial resistance of the air to be 100 pounds and varying as the square of the velocity, find the initial and final velocities, and the times of rising and falling.

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let R=radius of earth=20924640 feet,

$h=60.27R=1261128052.8$ feet=distance to moon,

$g=32.2$ feet, $W=600$ pounds, $\mu v^2=100$ pounds$=\frac{1}{6}W$,

t=time of ascent; t_1=time of descent, v=initial velocity, v_1=final velocity. Then $\mu=W/6v^2$; $1/k=\sqrt{(W/\mu)}=v\sqrt{6}$. $\therefore\ k=1/v\sqrt{6}$.

$h=(1/2gk^2)\log(1+v^2k^2)$, $t=(1/gk)\tan^{-1}vk$,

$t_1=(1/gk)\log[\sqrt{(1+v^2k^2)}+vk]$, $v_1=v/\sqrt{(1+v^2k^2)}$.

$\therefore\ (1+v^2k^2)=\frac{7}{6}$, $vk=1/\sqrt{6}$.

$\therefore\ v^2=gh/3\log(\frac{7}{6})$, $v=449657$ feet=85.16 miles per second.

$v_1=(\frac{6}{7})v=73$ miles per second.

$t=[v\sqrt{(6)}/g]\tan^{-1}(1/\sqrt{6})=13258.2$ seconds=3 hours, 40 minutes, 58.2 seconds.

$t_1=[v\sqrt{(6)}/g]\log\{[\sqrt{(7)}+1]/[\sqrt{6}]\}=13603.7$ seconds=3 hours, 46 minutes, 43.7 seconds.

In the above we have considered the resisting medium as extending to the moon.

57. Proposed by J. C. NAGLE, A. M., M. C. E., Professor of Civil Engineering, Agricultural and Mechanical College of Texas, College Station, Texas.

Over the intersection of two inclined planes slides a cord of uniform mass throughout its length. Find the equation of the path described by its center of gravity.

[No solution of this problem has been received. EDITOR.]

58. Proposed by **ALFRED HUME, C. E., D. Sc.,** Professor of Mathematics, University of Mississippi, University, Miss.

An endless uniform chain is hung over two small smooth pegs in the same horizontal line. Show that, when it is in a position of equilibrium, the ratio of the distance between the vertices of the two catenaries to half the length of the chain is the tangent of half the angle of inclination of the portions near the pegs. [From *Routh's Analytical Statics. Mathematicl Tripos, 1855.*]

I. Solution by **WILLIAM HOOVER, A. M., Ph. D.,** Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let h_1, h_2 be the vertical distances of the vertices of the two catenaries below the line $(=2a)$ joining the pegs ; then using the usual notation of the theory of the catenary, the origin being the middle of $2a$,

$$h_1+c_1-y=\tfrac{1}{2}c_1(e^{x/c_1}+e^{-(x/c_1)})\ldots\ldots(1),$$

$$s_1=\tfrac{1}{2}c_1(e^{x/c_1}-e^{-(x/c_1)})\ldots\ldots(2),$$

with like equations for the second curve.

From (1) and (2), α_1 being the inclination of the curve at the peg,

$$(dy/dx)=\tfrac{1}{2}(e^{x/c_1}-e^{-(x/c_1)})=\tan\alpha_1\ldots\ldots(3).$$

$$(ds_1/dx)=\tfrac{1}{2}(e^{x/c_1}+e^{-(x/c_1)})=\sec\alpha_1\ldots\ldots(4).$$

$$\therefore\ \sec\alpha_1+\tan\alpha_1=e^{a/c_1}\ldots\ldots(5).$$

$y=0$ in (1) gives $h_1+c_1=\frac{1}{2}c_1(e^{a/c_1}+e^{-(a/c_1)})=s_1/\sin\alpha_1\ldots\ldots(6)$,

by aid of (5), and by the usual theory, $c_1=s_1\cot\alpha_1\ldots\ldots(7)$, and then from (6) and (7),

$h_1=s_1\tan\frac{1}{2}\alpha_1\ldots(8)$. Similarly, $h_2=s_2\tan\frac{1}{2}\alpha_2\ldots(9)$, and $h_2+c_2=s_2/\sin\alpha_2\ldots(10)$.

For equilibrium, the right members of (6) and (10) are equal, since they measure the tensions in the two parts of the chain at either peg.

$$\therefore\ \frac{s_1}{s_2}=\frac{\sin\alpha_1}{\sin\alpha_2}=\frac{\cot\frac{1}{2}\alpha_2+\tan\frac{1}{2}\alpha_2}{\cot\frac{1}{2}\alpha_1+\tan\frac{1}{2}\alpha_1}\ldots\ldots(11).$$

Then finally,

$$\frac{h_1-h_2}{s_1+s_1}=\frac{(s_1/s_2)\tan\frac{1}{2}\alpha_1-\tan\frac{1}{2}\alpha_2}{(s_1/s_2+1}\ldots\ldots(12).$$

(11) in (12) gives

$$\frac{h_1-h_2}{s_1+s_1}=\tan\tfrac{1}{2}(\alpha_1-\alpha_2)\ldots\ldots(13).$$

II. Solution by **ALFRED HUME, C. E., D. Sc.,** Professor of Mathematics, University of Mississippi, P. O. University, Miss., and **HENRY HEATON, M. Sc.,** Atlantic, Iowa.

Let $2l$ be the length of the lower catenary, c its parameter, ψ the angle which its tangent at the right-hand peg makes with its directrix ; and let the same letters, accented, refer, similarly, to the upper catenary.

Then $\tan\psi=l/c$ and $\tan\psi'=l'/c'$, from which

$$\tan(\psi-\psi')=\frac{c'l-cl'}{cc'+ll'};\text{ and }\tan\tfrac{1}{2}(\psi-\psi')=\frac{\sqrt{(c^2+l^2)(c'^2+l'^2)}-cc'-ll'}{c'l-cl'}.$$

The tension in the chain at any point being equal to the weight of as much of it as would reach from that point to the directrix, and the resultant tensions in the two catenaries at the pegs being the same, it follows that the two catenaries have a common directrix. Also, the square of the ordinate of any point equals the sum of the squares of the parameter and the length of the arc between the vertex and that point. From these facts, $l^2+c^2=l'^2+c'^2$. Substituting,

$$\tan\tfrac{1}{2}(\psi-\psi')=\frac{c^2+l^2-cc'-ll'}{c'l-cl'}=\frac{(c-c')c+(l-l')l}{c'l-cl'}.$$

Remembering that $(c-c')(c+c')=(l'-l)(l+l')$, this may be reduced to $(c'-c)/(l+l')$.

But $c'-c$ is the distance between the vertices of the two catenaries, $l+l'$ is half the length of the chain, and $\psi-\psi'$ is the inclination of the portions near the peg. Hence the proposition is established.

Also solved by *G. B. M. ZERR*.

DIOPHANTINE ANALYSIS.

57. **Proposed by JOSIAH H. DRUMMOND, LL. D., Counselor at Law, Portland, Maine.**

Each of *five* of the digits may be the terminal figure of a perfect integral square. Each of *eighteen* combinations of two digits may be the *two* terminal figures of an integral square. Each of *one hundred and nineteen* combinations of three digits may be the *three* terminal figures of an integral square. *Under these conditions*, what is the greatest number of arrangements of the nine digits, all taken together, whose three terminal figures shall be those of a square number?

Solution by the PROPOSER.

The first two conditions are embraced in the third and must be disregarded: the terminal figure of the last three must be one of five digits; and the two terminal figures must be one of the eighteen combinations of two digits. There are 119 combinations of three digits that may be terminal figures: with each one of these combinations, within the terms of the question, the other six digits may be used as many times as there are combinations in the six taken together, that is $2\times3\times4\times5\times6=720$. Hence $119\times720=85{,}680$, the number required.

59. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.**

Find the sum of the mth power of all the numbers less than P and prime to it, and then by substitution find the sum when $m=1, 2, 3, 4, 5$.

Solution by the PROPOSER.

The sum of the mth powers of all numbers less than P and not prime to it is given by the sum of

$$\begin{aligned}
&a^m+(2a)^m+(3a)^m+\ldots\ldots+[(P/a).a]^m\\
&+b^m+(2b)^m+(3b)^m+\ldots\ldots+[(P/b).b]^m\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&-(ab)^m-(2ab)^m-(3ab)^m-\ldots\ldots-[(P/ab).ab]^m\\
&-(bc)^m-(2bc)^m-(3bc)^m-\ldots\ldots-[(P/bc).bc]^m\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+(abc)^m+(2abc)^m+(3abc)^m+\ldots\ldots+[(P/abc).abc]^m\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

$$=a^mS_{P/a}+b^mS_{P/b}+\ldots.-(ab)^mS_{P/ab}-(bc)^mS_{P/bc}-\ldots.+(abc)^mS_{P/abc}+\ldots.$$

$\therefore$ The sum of the mth powers of all the integers less than P and prime to it is

$$S_P-a^mS_{P/a}+b^mS_{P/b}-\ldots+(ab)^mS_{P/ab}+(bc)^mS_{P/bc}+\ldots-(abc)^mS_{P/abc}-\ldots. \quad (1).$$

But $S_P=\dfrac{P^{m+1}}{m+1}+\tfrac{1}{2}P^m+B_1\dfrac{m}{2\,!}P^{m-1}+B_3\dfrac{m(m-1)(m-2)}{4\,!}P^{m-3}$

$$+B_5\frac{m(m-1)(m-2)(m-3)(m-4)}{6\,!}P^{m-5}+\ldots\ldots$$

$$a^mS_{P/a}=\frac{P^{m+1}}{m+1}.(1/a)+\tfrac{1}{2}P^m+B_1\frac{m}{2\,!}P^{m-1}a+B_3\frac{m(m-1)(m-2)}{4\,!}P^{m-3}a^3$$

$$+B_5\frac{m(m-1)(m-2)(m-3)(m-4)}{6\,!}P^{m-5}a^5+\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Where $B_1=\frac{1}{6}$, $B_3=\frac{1}{30}$, $B_5=\frac{1}{42}$, $B_7=\frac{1}{30}$, etc., Bernoulli's numbers. Substituting the values of S_P, $a^mS_{P/a}$, $b^mS_{P/b}$, $(ab)^mS_{P/ab}$, etc., in (1) and letting S_m=the sum, we get

$$S_m=\frac{P^{m+1}}{m+1}\Big(1-(1/a)-(1/b)-(1/c)-\ldots.+(1/ab)+\ldots.\Big)$$

$$+\tfrac{1}{2}P^m\Big(1-n+\frac{n(n-1)}{2\,!}-\ldots.\Big)+B_1\frac{m}{2\,!}P^{m-1}\{1-a-b-c-\ldots.+ab+\ldots.\}$$

$$-B_3\frac{m(m-1)(m-2)}{4\,!}P^{m-3}\{1-a^3-b^3-c^3-\ldots.+a^3b^3+\ldots.\}$$

$$+B_5\frac{m(m-1)(m-2)(m-3)(m-4)}{6\,!}P^{m-5}\{1-a^5-b^5-c^5-\ldots.+a^5b^5+\ldots.\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Where n is the number of primes in P. $\therefore$ The coefficient of P^m is zero.

$$\therefore\ S_m = \frac{P^{m+1}}{m+1}[1-(1/a)][1-(1/b)][1-(1/c)]\ldots.$$

$$+\ B_1\frac{m}{2\,!}P^{m-1}(1-a)(1-b)(1-c)\ldots.$$

$$-\ B_3\frac{m(m-1)(m-2)}{4\,!}P^{m-3}(1-a^3)(1-b^3)(1-c^3)\ \ldots.$$

$$+\ B_5\frac{m(m-1)(m-2)(m-3)(m-4)}{6\,!}P^{m-5}(1-a^5)(1-b^5)(1-c^5)\ \ldots.$$

$$-\ldots\ldots+\ldots\ldots-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

When $m=1$, $S_1=\frac{1}{2}P^2[1-(1/a)][1-(1/b)][1-(1/c)]\ldots.$

When $m=2$, $S_2=\frac{1}{3}P^3[1-(1/a)][1-(1/b)][1-(1/c)]\ldots+\frac{1}{6}P(1-a)(1-b)(1-c)\ldots$

When $m=3$, $S_3=\frac{1}{4}P^4[1-(1/a)][1-(1/b)][1-(1/c)]\ldots+\frac{1}{4}P^2(1-a)(1-b)(1-c)\ldots$

When $m=4$, $S_4=\frac{1}{5}P^5[1-(1/a)][1-(1/b)][1-(1/c)]\ldots+\frac{1}{3}P^3(1-a)(1-b)(1-c)\ldots$
$-\frac{1}{30}P(1-a^3)(1-b^3)(1-c^3)\ldots.$

When $m=5$, $S_5=\frac{1}{6}P^6[1-(1/a)][1-(1/b)][1-(1/c)]\ldots+\frac{5}{12}P^4(1-a)(1-b)(1-c)\ldots$
$-\frac{1}{12}P^2(1-a^3)(1-b^3)(1-c^3)\ldots.$

When $m=6$, $S_5=\frac{1}{7}P^7[1-(1/a)][1-(1/b)][1-(1/c)]\ldots+\frac{1}{2}P^3(1-a)(1-b)(1-c)\ldots$
$-\frac{1}{6}P^3(1-a^3)(1-b^3)(1-c^3)\ldots.\frac{1}{42}P(1-a^5)(1-b^5)(1-c^5)\ldots.$

PROBLEMS FOR SOLUTION.

ALGEBRA.

83. Proposed by J. MARCUS BOORMAN, Consultative Mechanician, Counselor at Law, Inventor, Etc., Woodmere, Long Island, N. Y.

Solve $x^2+y=8\ldots\ldots(1)$; $y^2+x=69\ldots\ldots(2)$, true to four decimals.

84. Proposed by BENJ. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, O.

On the present electoral basis, if all the electoral votes of each State are cast solid for one or the other of two presidential candidates, how many conbinations of States are possible for a total of 273 votes for the winning candidate ?

*** Solutions of these problems should be sent to J. M. Colaw, not later than May 10.

GEOMETRY.

91. Proposed by LEONARD E. DICKSON, Ph. D., Instructor in Mathematics in the University of California, Berkeley, Cal.

If a point A remain fixed while a point B moves along a given straight line, prove that the locus of the vertex C of the triangle ABC, similar to a given triangle and lying always on the same side of AB, is a straight line. Verify geometrically for the case in which the angles at A and C remain equal.

92. Proposed by **JOSIAH H. DRUMMOND, LL. D., Counselor at Law, Portland, Maine.**

Let $ABCD$ be a quadrilateral inscribed in a circle. Draw the diagonals AC and BD. Show that $AB.BC : DC.AD=BD : AC$. [From a note in *Young's Geometry*, edition of 1830.]

93. Proposed by **G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.**

While surveying in a level field I notice a mountain behind a hill. Wishing to know the height of each I take the angles of elevation of the tops of both and find them to be $\beta=45°$, $\delta=40°$, I then measure a straight line $a=400$ feet and find the angles of elevation of the tops to be $\gamma=42°$, $\mu=38°$. After measuring $b=300$ feet more in the same straight line I find the elevations to be $\lambda=40°$, $\nu=36°$. Find the height of each.

*** Solutions of these problems should be sent to B. F. Finkel, not later than May 10.

CALCULUS.

73. Proposed by **MOSES COBB STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Ind.**

$$\text{Solve}\int_0^{\frac{1}{4}\pi}\log(1-\tan x)dx$$

74. Proposed by **EDWARD R. ROBBINS, A. B., Mathematical Master in the Lawrenceville School, Lawrenceville, N. J.**

A circular ring, whose radii are a and b, is cut by a plane making the area of the section (or sections) a maximum. Required the position of the plane, and the nature and area of the section (or sections).

*** Solutions of these problems should be sent to J. M. Colaw, not later than May 10.

DIOPHANTINE ANALYSIS.

64. Proposed by **JOHN M. COLAW, A. M., Monterey, Va.**

Find two cubic proper fractions whose product is a square proper fraction. Can a *general* solution be made?

65. Proposed by **F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.**

Find (1) four consecutive numbers whose sum is a square, and (2) four consecutive numbers the sum of whose squares is a square.

*** Solutions of these problems should be sent to J. M. Colaw, not later than May 10.

MISCELLANEOUS.

60. Proposed by **G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

A tube of uniform cross section, small compared with its length, is bent into the form of a cycloid, its open ends lying at the cusps, and this cycloid is placed with its axis vertical and its vertex downwards. Equal quantities of fluids of specific gravity σ_1 and σ_2 are

poured in at the two cusps, the quantity of each being such as would fill a length of the tube equal to its axis a. If the fluids do not mix, find the distance x_1, x_2 of the upper levels of the fluids from the vertex measured along the cycloidal arc. [From *Proctor's Geometry of the Cycloid.*]

61. **Proposed by F. M. SHIELDS, County Surveyor, Coopwood, Miss.**

Of three chronometers, A, B, and C, A keeps true time; B gains 5 minutes and $15\frac{3}{4}$ seconds a day by true time, and C loses 7 minutes and 15 3-7 seconds a day by true time. The hands of all three watches are set at 12 noon on a certain day. What is the time by the true watch, A, on the *fifth* day after time when the hands of the fast watch, B, point to 12, and what is the time by the true watch, A, on the *tenth* day after time when the hands of the slow watch, C, point to 12?

*** Solutions of these problems should be sent to J. M. Colaw, not later than May 10.

PERIODICALS.

The Open Court. A Monthly Magazine. Devoted to Science of Religion, the Religion of Science, and the extension of the Religious Parliament Idea. Edited by Dr. Paul Carus; Assistant Editor, T. J. McCormack and Associate Editors E. C. Hegeler and Mary Carus. Price, $1.00 per year in advance. Single copies, 10 cents.

The April *Open Còurt* presents its readers as a frontispiece with a unique specimen of Japanese art in the shape of a brilliantly colored poster called *The Pure Land* or *Western Paradise*, by a famous Tokyo artist, Mishima. The posters were made especially for the *Open Court*, and the sheets imported from Japan. The conception is delicate, and the tints little short of exquisite.

In the leading article Dr. Woods Hutchinson, of Buffalo discourses upon Courage the Chief Virtue, which is opposed by the author to meekness and submissiveness usually taught, and contrasts the sublime fortitude of Christ to the cowardice of many interpretations of his religion. Dr. Moncure D. Conway continues his series of articles on Solomonic Literature, giving in the present number the mythological interpretation of the traditions connected with the wives of Solomon. The illustraded article of the number is by Dr. Paul Carus on the Human Heart as Mirrored in Religious Art. Old wood cuts are reproduced, portraying the various conceptions of the soul as the vehicle of good and evil, while representations of modern ideas are also given.

Captain Pfoundes of Japan writes entertainingly of the present situation in China, and his article is accompanied by a handsome half tone illustration of the Seven Sages of the Bamboo-Grove. Mr. Sandison of Glasgow reports the Gifford Lectures now being delivered in Scotland by Dr. Bruce; and Mr. Edmund Noble of Boston gives us Some Parallels Between Theology and Science. Prof. I. W. Howerth of the University of Chicago treats of the crying problems of social life. Poems, with a new hymn by Dr. Paul Carus, and a number of book reviews complete the number.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

The Cuban crisis naturally demands more space in the editorial department of the *American Montly Review of Reviews* than any other single topic. The whole matter is reviewed in the light of the latest and most authentic information received up to the time of going to press. The *Review* is convinced that the country desires and will demand intervention in Cuba, that the real question at issue is the relief of Cuba, not the sattlement of the *Maine* incident, and that Spain's final withdrawal from the Western hemisphere will be the only satisfactory termination of the present trouble.

The Monist. A Quarterly Magazine devoted to the Philosophy of Science. Edited by Dr. Paul Carus; T. J. McCormack, Assistant Editor; E. C. Hegeler, and Mary Carus, Associate Editors. Price, $2.00 per year in advance. Single number, 50 cents. The Open Court Publishing Co., Chicago, Ill.

Prof. Ferdinand Hüppe, the well known Professor of Hygiene in the University of Prague, contributes an interesting and important article to the April *Monist* on the Causes of Infectious Diseases. Prof. Hüppe is a bacteriologist of the modern school but nevertheless opposes the main doctrines of Koch, Pasteur and Virchow, and few will dissent from the reasonableness of the position he takes, which harmonizes the facts of the new theories with the established principles of the old. Both physicians and laymen will be interested in Prof. Hüppe's presentation of modern bacteriology.

In the same number, the famous Italian criminologist, Prof. Cesare Lombroso seeks to establish his favorite theory of the degeneracy of genius by considering certain regressive phenomena in evolution. Dr. Woods Hutchinson writes passionately and with rare ability upon Lebenslust, or the joy of life. A distinguished English lady, E. E. Constance Jones, discusses An Aspect of Attention. Prof. John Dewey, of the University of Chicago, discusses ethics in the light of evolution. And, finally, the editor, Dr. Paul Carus, in a long article on the Unmateriality of Soul and God, seeks to lay a firm foundation for correct views of these momentous questions. The number concludes with entertaining Literary Correspondence from Europe, and the usual number of Book Reviews in the field of science, philosophy and religion.

SOME ERRATA IN FEBRUARY NUMBER.

Page 42, line 34, for "defective" read defect.

Page 43, line 12, for "$-xy=6$" read $-xy=-6$.

Page 44, line 17, for last "$\pm_1$" read $\mp_1$; line 30, omit "(" before xy.

Page 47, line 18, for "PN" read DN; line 19, supply "(" before $FG+NE$).

Page 48, line 7, for "$12\pi n+\pi n^2/1200=100$" read $12\pi n+\pi n^2/100=1200$.

Page 52, line 28, for "$3(6^2-1^1)$" read $3(6^2-1^2)$.

Page 53, line 3, after 208, insert 608; line 33, for "13" read 15.

Page 54, line 3, for "363" read 368.

Page 59, line 21, for "$7m^2n$" read $7m^3n$.

Page 60, line 1 of problem 70, for "π/n" read $\pi/2n$; and in last line, for "$\pi/2l$" read $\pi/2n$.

Page 61, problem 63, for "x^2" read x^3, and for "105498" read 105489.

Page 66, line 13 from bottom, for "Page 22" read Page 17.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. APRIL, 1898. No. 4.

ON SOME CASES WHEN THE QUINTIC IS SOLVABLE BY ELEMENTARY METHODS.

By A. C. BURNHAM, University of Illinois.

Abel has shown that the general equation of the fifth degree is not solvable by the ordinary method of extraction of roots. Abelian equations and cyclical equations in which the *roots* satisfy a certain condition can be so solved. It would be of value to have the necessary and sufficient condition which must be satisfied by the *coefficients*, instead of by the roots, in order that the given equation be an Abelian, or a cyclical, or, in fact, be solvable by root extraction, so that one could immediately tell, given an equation, whether it were solvable. I believe no one has as yet deduced such a condition for the general quintic.

It is the object of this paper to give, not the general condition indeed, but one which covers many special cases, in the hope that it may prove of interest or of value in further study of the subject.

Let the general quintic be

$$f(x)=x^5+a_1x^4+a_2x^3+a_3x^2+a_4x+a_5=0 \ldots\ldots\ldots\ldots\ldots (\mathrm{I}),$$

and let it be divisible without a remainder by

$$\varphi(x)=x^2+px+q=0.$$

Now divide $f(x)$ by $\varphi(x)$ and let the result be

$$\frac{f(x)}{\varphi(x)}=\psi(x)+\frac{F_1(p,\ q)x+F_2(p,\ q)}{\varphi(x)}$$

where

$$\psi(x) = x^3 + A_1 x^2 + A_2 x + A_3 = 0,$$

A_1, A_2 and A_3 being functions of p, q, a_1, a_2, and a_3, and where F_1 and F_2 arranged according to q are

$$F_1 = q^2(a_1 - 2p) + q(-a_3 + a_2 p - a_1 p^2 + p^3) + a_5 = 0 \;\ldots\ldots\ldots (\mathrm{II})$$

and

$$F_2 = q^2 + q(-a_2 + 2a_1 p - 3p^2) + a_4 - a_3 p + a_2 p^2 - a_1 p^3 + p^4 = 0 \ldots . (\mathrm{III}).$$

Now if (II) and (III) were considered simultaneous and solved for p and q, they would give all the ten values each of p and q which would make $\varphi = 0$ divide $f(x) = 0$ without a remainder, but this would involve the solution of an equation of the tenth degree.

If, however, (II) and (III) be considered identical and the absolute terms and the coefficients of q in the two be equated after dividing (II) by $a_1 - 2p$, there results two equations in p, which must possess at least one common root.

These two equations in p are

$$5p^3 - 6a_1 p^2 + (2a_1^2 + a_2)p + a_3 - a_1 a_2 = 0 \ldots\ldots\ldots\ldots\ldots (\mathrm{IV}),$$

and

$$2p^5 - 3a_1 p^4 + (2a_2 + a_1^2)p^3 - (2a_3 + a_1 a_2)p^2 + (2a_4 + a_1 a_3)p + a_5 - a_1 a_4 = 0 \ldots (\mathrm{V}).$$

These equations (IV) and (V) will have a common root when their resultant, R, formed by Sylvester's method, is equal to zero, that is, when

$$R = \begin{pmatrix} 5 & -6a_1 & 2a_1^2 + a_2 & a_3 - a_1 a_2 & 0 & 0 & 0 & 0 \\ 0 & 5 & -6a_1 & 2a_1^2 + a^2 & a_3 - a_1 a_2 & 0 & 0 & 0 \\ 0 & 0 & 5 & -6a_1 & 2a_1^2 + a_2 & a_3 - a_1 a_2 & 0 & 0 \\ 0 & 0 & 0 & 5 & -6a_1 & 2a_1^2 + a_2 & a_3 - a_1 a_2 & 0 \\ 0 & 0 & 0 & 0 & 5 & -6a_1 & 2a_1^2 + a_2 & a_3 - a_1 a_2 \\ 2 & -3a_1 & a_1^2 + 2a_2 & -2a_3 - a_2 a_1 & 2a_4 + a_1 a_3 & a_5 - a_1 a_4 & 0 & 0 \\ 0 & 2 & -3a_1 & a_1^2 + 2a_2 & -2a_3 - a_1 a_2 & 2a_4 + a_1 a_3 & a_5 - a_1 a_4 & 0 \\ 0 & 0 & 2 & -3a_1 & a_1^2 + 2a_2 & --2a_3 - a_1 a_2 & 2a_4 + a_1 a_3 & a_5 - a_1 a_4 \end{pmatrix}$$

$$= 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{VI}).$$

This is therefore the condition that the general quintic be solvable by considering (II) and (III) identical.

The value of the common root p is easily found by differentiating R, and this value of p substituted in (II) or (III) gives in general two values of q corresponding. The value of p common to (IV) and (V) with either of the corresponding values of q substituted in $\varphi = 0 = x^2 + px + q$ gives two roots of the proposed equation (I). Two more roots are given by taking the same value of p with the

other corresponding value of q.

We see that the condition $R=0$ among the coefficients requires that, for some one value of p, there shall be two values of q, *i. e.* that, for example, there exist among the roots such a relation as

$$x_1+x_3=x_2+x_4$$

with

$$x_1x_3=x_2x_4,$$

where x_0, x_1, x_2, x_3, x_4 represent the five roots. The other root, the fifth, is then given by $x_0=2p-a_1$.

In case of equal roots, so that for a known value of p the two values of q are equal, or in any case when it is desirable or easier, three of the roots can be found from $\psi(x)=0$ when any pair of corresponding values of p and q are known.

In this discussion no restriction is placed upon the character of the values of a_1, a_2, ... a_5 or p or q. They may be complex.

Whenever, therefore, the coefficients of a quintic satisfy the condition (VI), then p can have at most only nine values (instead of ten as in the general case) and the equation is solvable by this method. q may at the same time have ten values.

Example: Let the given equation be

$$x^5-10x^4+17x^3+76x^2-228x+144=0.$$

In this case the condition (VI) is satisfied and equations (IV) and (V) have a single common root, $p=-3$. Then from (II) or (III), we have

$$q^2+16q-36=0,$$

from which $q=2$ or -18. Therefore

$$x^2-3x+2=0, \text{ from which } x=1 \text{ or } 2.$$

$$\text{or } x^2-3x-18=0, \text{ from which } x=-3 \text{ or } 6,$$

and finally $x=2p-a_1=-6+10=4$, and the roots are 1, 2, -3, 4 and 6.

Urbana, Ill., December 27, 1897.

NOTE ON THE PRACTICAL APPLICATION OF A SUBSTITUTION GROUP IN SPHERICAL TRIGONOMETRY.

By DR. G. A. MILLER.

Several years ago Professor Study published a very valuable article* on spherical trigonometry in which he demonstrated the fertility of the group concept in this branch of elementary mathematics. His efforts were directed toward making the theory more general and more complete, and he explicitly states that the practical geometer would not find any results in his article which he could directly employ.

In what follows, our aim, on the contrary, is entirely practical. It is our belief that a fair knowledge of spherical trigonometry can be obtained more quickly by devoting one lesson to the substitution group given below than would otherwise be possible.

We let a, b, c represent the three sides of a spherical triangle, and α, β, γ the supplements of the angles opposite these sides, taken in order. From the polar triangle we have that the substitution

$$a\alpha.b\beta.c\gamma \dagger$$

will transform any formula into one which is equally true. It is also evident that any formula which has been derived without assigning any particular value to any one of the six parts is transformed into a formula which is equally true by means of the substitutions

$$ab.\alpha\beta \qquad bc.\beta\gamma.$$

Hence such a formula is transformed into one that is equally true if these substitutions are applied successively. In other words, any substitution of the group generated by the given three substitutions transforms any such formula into one that is equally true. This substitution group (G) is well known. It is composed of the following substitutions :

1	
$abc.\alpha\beta\gamma$	$a\alpha.b\beta.c\gamma$
$acb.\alpha\gamma\beta$	$a\beta c\alpha b\gamma$
$ab.\alpha\beta$	$a\gamma b\alpha c\beta$
$ac.\alpha\gamma$	$a\beta.b\alpha.c\gamma$
$bc.\beta\gamma$	$a\gamma.b\beta.c\alpha$
	$a\alpha.b\gamma.c\beta$

If a general formula like

*Study, *Saechsischen Abhandlungen*, 20, pp. 87—231.

†This means that a is replaced by α and that α is replaced by a, etc.

$$\sin a\cos\beta+\cos b\sin c+\sin b\cos c\cos\alpha=0$$

is not transformed into itself by any one of the substitutions of G besides unity, we obtain 12 and only 12 formulae from it by transforming it by the substitutions of G. From each one of these 12 formulae we may obtain the other 11 by applying to it the substitutions of G which differ from unity. These 12 formulae thus form a closed system in regard to G. It is of interest to observe the closed systems in regard to the different subgroups of G and to interpret them geometrically. When a general formula like

$$\cos a=\cos b\cos c-\sin b\sin c\cos\alpha$$

is transformed into itself by one substitution of G which differs from unity $(bc.\beta\gamma)$ we may obtain 6 and only 6 formulae from it by transforming it by all the substitutions of G, and from each one of these 6 we may obtain the other 5 in a similar manner. Each of these six formulae may be obtained from any given one of them by transforming it by means of either one of two substitutions of G.

In general, if such a formula is transformed into itself by just k substitutions of G we may obtain $12\div k$ formulae from it by transforming it by the substitutions of G, and all of these can be obtained from any one of them by transforming it by means of the substitutions of G, k substitutions giving the same formula in each case. It is clear that k must be a divisor of 12; *i. e.* the number of the substitutions of G that transform any general formula of the spherical triangle into itself is a divisor of 12.

If we replace α, β, γ by A, B, C respectively, the first half of the substitutions of G form a subgroup which may be employed in a similar manner to derive the formulae of the plane oblique triangle from each other. Hence this group might be made a very useful and general aid to remember the formulae of trigonometry. The explicit introduction of G into the text-books on spherical trigonometry would also furnish the means of making the student thoroughly acquainted with some of the elements of a concept of great fertility.

Cornell University, March, 1898.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

87. **Proposed by E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.**

A and B set out from the same place, and in the same direction. A travels uniform-

ly 18 miles per day, and after 9 days turns and goes back as far as B has traveled during those 9 days; he then turns again, and, pursuing his journey, overtakes B $22\frac{1}{2}$ days after the time they first set out. It is required to find the rate at which B uniformly traveled. [From *Greenleaf's Arithmetic.*]

Solution by J. F. TRAVIS, Student in the Ohio State University, Columbus, Ohio.

Let $\frac{2}{2}$=number of miles B traveled per day. Then

$22\frac{1}{2}\times\frac{2}{2}$=total distance B traveled.

$18\times9=162$=number of miles traveled by A in 9 days, and

$162-9\times\frac{2}{2}$=number of miles A is from starting point.

$(9\times\frac{2}{2})\div18=\frac{2}{2}\div2$=number of days A traveled backwards.

$\therefore\ 9+\frac{2}{2}\div2$=total number of days A traveled.

$22\frac{1}{2}-(9+\frac{2}{2}\div2)=13\frac{1}{2}-\frac{2}{2}\div2$=number of days in which A must overtake B.

To overtake B, A must travel $[22\frac{1}{2}\times\frac{2}{2}-(162-9\times\frac{2}{2})]\div18$ days.

$\therefore\ [22\frac{1}{2}\times\frac{2}{2}-(162-9\times\frac{2}{2})]\div18=13\frac{1}{2}-\frac{2}{2}\div2$, from which we find $\frac{2}{2}=10$= number of days.

This problem was also solved by *F. R. HONEY*, *G. B. M. ZERR*, and *M. A. GRUBER*. Mr. Gruber gave an alegebraic solution and discussed the problem in general.

88. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Find the principal of a note given March 19, 1891, bearing interest 6%. Payments: September 1, 1892, \$243.50; January 19, 1893, \$6.90; April 13, 1894, \$19.10; September 19, 1894, \$110.90. Amount due February 22, 1897, \$229.10.

Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; P. S. BERG, B. Sc., Superintendent of Schools, Larimore, N. D., and NELSON L. RORAY, Professor of Mathematics, South Jersey Institute, Bridgeton, N. J.

The amount \$229.10 due February 22, 1897, has been running 2 years, 5 months, 3 days. Hence the principal is easily found to be \$200.

The payments of \$6.90 and \$19.10 are evidently less than the interest.

Hence the \$200 has been running since September 1st, 1892, or 2 years, 18 days. But the payments \$6.90, \$19.10 and \$110.90 must be added to the \$200, making \$336.90.

Working back we find that this on September 1st, 1892, was \$300. But \$243.50 was paid on this date; hence we have \$543.50 as principal and interest which has been running since March 19, 1891, or 1 year, 5 months, 12 days.

Whence amount=\$543.50. Rate=6%. Time=1 year, 5 months, 12 days, and hence principal is easily found to be \$500.

Also solved by *G. B. M. ZERR*.

89. Proposed by NELSON L. RORAY, South Jersey Institute, Bridgeton, N. J.

Solve by pure arithmetic. A criminal having escaped from prison traveled 10 hours before his escape was known; he was then pursued so as to be gained upon 3 miles an hour: after his pursuers had traveled 8 hours they met an express going at same rate as themselves, who had met the criminal 2 hours and 24 minutes before; in what time from the commencement of the pursuit will they overtake him?

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; MARTIN SPINX, Wilmington, O.; F. R. HONEY, Ph. B., New Haven, Conn.; ALBERT J. GIBBS, Salida, Col.; and AMELIA BACH, Salida, Col.

When the pursuers met the express they had been in pursuit 8 hours. When the express met the criminal, the pursuers had been following the criminal $8-2\frac{2}{5}=5\frac{3}{5}$ hours, and the criminal had been escaping for $10+5\frac{3}{5}=15\frac{3}{5}$ hours.

As the express and the pursuers traveled at the same rate, the distance traveled by the criminal in $15\frac{3}{5}$ hours was traveled by the pursuers in $8+2\frac{2}{5}=10\frac{2}{5}$ hours. The pursuers, in this time, gained $10\frac{2}{5}\times3=31\frac{1}{5}$ miles. This distance was evidently traveled by the criminal in $15\frac{3}{5}-10\frac{2}{5}=5\frac{1}{5}$ hours.

$\therefore$ The criminal's rate of travel was $31\frac{1}{5}\div5\frac{1}{5}=6$ miles per hour.

The crimiual therefore had the start of $10\times6=60$ miles.

But the pursuers gained 3 miles per hour. Then, to gain 1 mile they had to travel $\frac{1}{3}$ hour, and to gain the 60 miles they had to travel $60\times\frac{1}{3}=20$ hours= the time required.

Also solved by *J. H. DRUMMOND, WILL RYAN, D. G. DORRANCE, Jr., W. H. DRANE, G. B. M. ZERR, FREMONT CRANE, M. E. GRABER, B. F. YANNEY*, and *J. A. MOORE.*

ALGEBRA.

81. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Show that
$$\frac{a_1^r}{(a_1-a_2)(a_1-a_3)(a_1-a_4)\ldots(a_1-a_n)}$$
$$+\frac{a_2^r}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)}+\ldots\frac{a_n^r}{(a_n-a_1)(a_n-a_2)\ldots(a_n-a_{n-1})}$$
is zero if r is less than $n-1$; to 1 if $r=n-1$, and to $a_1+a_2+a_3+\ldots a_n$ if $r=n$.

[*C. Smith's Treatise on Algebra*, Ex. 53, page 104.]

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

The fractions being reduced to their least common denominator, every term of the numerator contains the factor a_1-a_2 except the first and the second. If, in the numerator, we put $a_1=a_2$, the first two terms become the same with opposite signs and each of the remaining terms has a zero-factor. Hence the numerator vanishes under this supposition, and, therefore, a_1-a_2 is a factor of it. Similarly every factor of the denominator may be shown to be a factor of the numerator. Now the latter is a homogeneous expression of a degree less than that of the denominator by $n-1-r$, there being $n-1$ factors in the denominator of each of the origiñal fractions.

If $r<n-1$, the numerator is of lower degree than the denominator. But, as proved above, there are as many conditions that cause the numerator to vanish as there are factors in the denominator. In this case the number of these is greater than the degree of the numerator, which is, therefore, identically equal to zero.

If $r=n-1$, the numerator and the denominator are of equal degree, and, being composed of the same factors, the fraction equals 1.

If $r=n$, the degree of the numerator is one greater than that of the denominator. Hence, besides the factors common to both, there must be in the numerator one other factor of the first degree. Since this factor must be symmetrical with reference to a_1, a_2, a_3, etc., it is $a_1+a_2+a_3+\ldots a_n$.

This last is, therefore, the value of the fraction, the numerical coefficient independent of a_1, a_2, a_3, etc., evidently being unity.

Also solved by *C. W. M. BLACK*.

82. **Proposed by B. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and F. P. MATZ, D. Sc., Ph.D. Mechanicsburg, Pa.**

$$\left.\begin{array}{l} y^2+yz+z^2=a^2 \\ z^2+zx+x^2=b^2 \\ x^2+xy+y^2=c^2 \end{array}\right\} \text{find } x,\ y, \text{ and } z.$$

[*C. Smith's Treatise on Algebra*, Ex. 31, page 172.]

I. **Solution by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.**

From (1)+(2)+(3), $2(y^2+z^2+x^2)+yz+zx+xy=a^2+b^2+c^2 \ldots\ldots\ldots(4)$.

Squaring (4)

$4(y^2+z^2+x^2)^2+4(y^2+z^2+x^2)(yz+zx+xy)+(yz+zx+xy)^2=(a^2+b^2+c^2)^2 \ldots(5)$.

From $2(1)^2+2(2)^2+2(3)$

$4(y^2+z^2+x^2)^2+4(y^2+z^2+x^2)(yz+zx+xy)-2(yz+zx+xy)^2=2(a^4+b^4+c^2) \ldots(6)$.

From $\sqrt{\{[(5)-(6)]/3\}}$

$yz+zx+xy=\pm\sqrt{\{\frac{1}{3}[(2a^2b^2+2b^2c^2+2a^2c^2)-(a^4+b^4+c^4)]\}} \ldots\ldots\ldots\ldots(7)$.

Put second member$=m$; then from $\sqrt{[6(7)+2(4)]}$

$2(y+z+x)=\pm\sqrt{[2(a^2+b^2+c^2)+6m]} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$.

From (4)+(7)−2(2) $2y(y+z+x)=a^2-b^2+c^2+m. \ldots\ldots\ldots\ldots(9)$.

From (9)÷(8) and restoring m

$$y=\frac{a^2-b^2+c^2\pm\frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}}}.$$

Similarly, $$z=\frac{a^2+b^2-c^2\pm\frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[12a^2b^2-3(a^2+b^2+c^2)^2]}\}}},$$

and $$x=\frac{b^2+c^2-a^2\pm\frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\pm\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)]}\}}}$$

II. **Solution by J. SCHEFFER, A. M., Hagerstown, Md.**

Subtracting (1) from (2) we get $(x-y)(x+y+z)=b^2-a^2$, or putting $x+y+z=s$, $(x-y)s=b^2-a^2 \ldots\ldots(4)$, and subtracting (1) from (3), we thus get $(x-z)s=c^2-a^2 \ldots(5)$. From (4) and (5) we obtain $y=(sx+a^2-b^2)/s \ldots(6)$, and $z=(sx+a^2-c^2)/s \ldots\ldots(7)$. Adding x to both members of (6) and (7), we have $x+y+z=(3sx+2a^2-b^2-c^2)/s$, or $s^2=3sx+2a^2-b^2-c^2 \ldots\ldots(8)$, whence

$x=(s^2+b^2+c^2-2a^2)/3s\ldots\ldots(9)$, and then by (6) and (7), $y=(s^2+a^2-2b^2+c^2)/3s\ldots\ldots(10)$; $z=(s^2+a^2+b^2-2c^2)/3s\ldots\ldots(11)$.

Substituting in (3), expanding, and arranging, we obtain,

$s^4-(a^2+b^2+c^2)s^2=a^2b^2+a^2c^2+b^2c^2-a^4-b^4-c^4\ \ldots\ldots\ldots\ldots\ldots\ldots(12)$, whence $s^2=\frac{1}{2}[a^2+b^2+c^2\pm\sqrt{3}.\sqrt{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}]$, or putting the irrational part$=S$, so that $S=\sqrt{3}.\sqrt{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}\ldots\ldots(13)$. $s^2=\frac{1}{2}(a^2+b^2+c^2\pm S)\ldots\ldots(14)$.

Substituting now in (9), (10), (11), we obtain

$$x=[3(b^2+c^2-a^2)\pm S]/6s\ ;$$
$$y=[3(a^2+c^2-b^2)\pm S]/6s\ ;$$
$$z=[3(a^2+b^2-c^2)\pm S]/6s.$$

(13), (14), and (15) suffice to determine the values of x, y, and z.

III. **Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

(1)−(2) gives $(y-x)(x+y+z)=a^2-b^2\ldots\ldots(4)$.

(1)−(3) gives $(z-x)(x+y+z)=a^2-c^2\ \ldots\ldots(5)$.

(4)÷(5) gives $y=[(b^2-c^2)x+(a^2-b^2)z]/[a^2-c^2]\ldots\ldots(6)$.

(6) in (3) gives $(a^4+b^4+3c^4+a^2b^2-3a^2c^2-3b^2c^2)x^2+(a^4+a^2b^2+3b^2c^2-2b^4-3a^2c^2)xz+(a^4-2a^2b^2+b^4)z^2=a^4c^2-2a^2c^4+c^6\ldots\ldots(7)$.

Let $z=vx$ in (2) and (7), and divide (7) by (2),

$$(a^4b^2-2a^2b^4+b^6-a^4c^2+2a^2c^4-c^6)v^2+(a^4b^2-2b^6+3b^4c^2+a^2b^4-3a^2b^2c^2-a^4c^2+2a^2c^4-c^6)v=a^4c^2+2a^2c^4-c^6-a^4b^2-b^6-3b^2c^4-a^2b^4+3a^2b^2c^2-3b^4c^2,$$

or $v^2+2Av=B$, suppose. $\therefore\ v=\pm\sqrt{(A^2+B)}-A$.

$$x=\frac{b}{\sqrt{(1+v+v^2)}}=\frac{b^2+c^2-a^2\pm\frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\sqrt{\{2(a^2+b^2+c^2)\pm2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}}}.$$

$\therefore\ x=(b^2+c^2-a^2\pm C)/D$, suppose.

$y=(a^2+c^2-b^2\pm C)/D$.

$z=(a^2+b^2-c^2\pm C)/D$.

Prof. Cooper D. Schmitt remarks that nine different solutions of this problem can be found in the *Mathematical Magazine*, Vol. II, Nos. 8 and 10, on pages 141 and 193.

Prof. Zerr sent three solutions.

GEOMETRY.

85. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

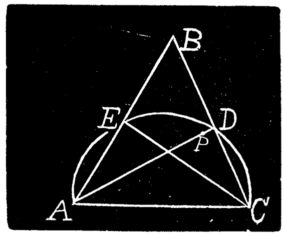
Prove by pure geometry. Give direct proof, if possible.

If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

[From *Wentworth's Plane Geometry*, exercise 43, page 72.]

I. Solution by J. M. COLAW, A. M., Monterey, Va., and EDMUND FISH, Hillsboro, Ill.

In triangle ABC, let AD, CE, be the bisectors of angles A and C, and $AD=CE$. Then angle A=angle C, and triangle ABC is isosceles.



Suppose a circle passed through A, C, and E. It will also pass through D. If not, suppose that it cut AD in any point P short of D. Then arc $EB>$ arc PE, since $\angle ECA(=\angle DCE)>\angle PCE$.

Also, arc PE=arc PC, since $\angle EAP=\angle PAC$. Whence arc $AEP>$arc EPC.

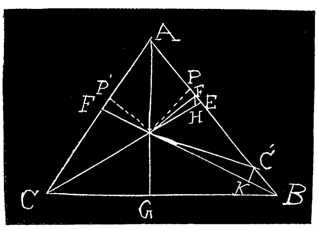
$\therefore$ chord $BP>$chord CE. But by hypothesis, $AD=CE$. $\therefore$ $AP>AD$, which is absurd. In the same way it may be shown that the supposition that the circle, which passes through A, C, and E, cuts BD in any point P', beyond D, also leads to an absurdity. The circle must therefore pass through D. Hence $\angle EAD(=\frac{1}{2}\angle A)=\angle DCE(=\frac{1}{2}\angle C)$.

$\therefore$ $\angle A=\angle C$, and triangle ABC is isosceles.

II. Solution by OTTO CLAYTON, Teacher of Mathematics and Physics, Remington High School, Remington, Ind.

Draw the bisector AG meeting the given bisectors CE and BF of the given triangle ABC, in the point O. Revolve ACG about AG as an axis until AC coincides with AB. Then the point C will fall within the segment AB, on the point B, or without the segment AB, according as $\angle ACO$ is greater, equal to, or less than $\angle ABO$. And the point F will fall within AE, on the point E or within EB, according as $\angle ACO$ is greater, equal to, or less than $\angle ABO$.

But $\angle ACO$ cannot be greater than ABO, for $(C'O+OE)=CE$ would be less than $(BO+OF')=BF$, which is contrary to the hypothesis that $CE=BF$.



Likewise $\angle ACO$ can not be less than $\angle ABO$, for $(B'O+OE)=CE$ would be greater than $(BO+OF)$ $=BF$, which is contrary to the hypothesis. Therefore $\angle ACO=\angle ABO$, and C falls upon B. Therefore the triangle is isosceles.

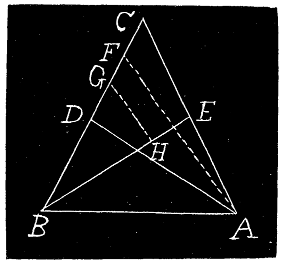
I think this is a simple proof and does not involve anything outside of the first book of Wentworth.

In the proof I did not show how $OE+OC'$ is less than $BO+OF'$. When the perpendicular OP falls between OE and OB the reason is obvious. When OP falls without, construct equilateral triangles $F'OH$ and $C'OK$. Then prove HE less than KB.

III. **Solution by the PROPOSER.**

Let the bisectors BE and AD be equal, prove triangle ABC isosceles.

Three suppositions are possible.



1st, $\angle A > \angle B$; 2nd, $\angle A < \angle B$; 3rd, $\angle A = \angle B$.

First, suppose $\angle A > \angle B$: then $\frac{1}{2}\angle A > \frac{1}{2}\angle B$. Construct $\angle FAD = \angle CBE$. Then in the triangle FAB, $FB > FA$ (greater side opposite greater angle). Lay off on BF a distance BG equal to AF, and draw GH parallel to FA. Then the triangle BGH=triangle FAD ($BG = FA$, by construction, $\angle GBH = \angle FAD$, for the same reason, $\angle BGH = \angle DFA$, exterior-interior angles.)

$\therefore$ $DH = BA$ (homologous sides of equal triangles) *which is absurd*, because $BE = AD$, by hypothesis, and BH is only a part of BE.

Second, in a similar manner it can be shown that $\angle B$ cannot be greater than A; *i. e.* $\angle A$ cannot be less than $\angle B$.

Third, as $\angle A$ can neither be greater nor less than $\angle B$, it must be equal to $\angle B$. $\therefore$ the triangle is isosceles. Q. E. D.

For other demonstrations of this problem, see Vol. II., pages 158, 189—192. EDITOR.

86. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

Prove that the four conics which have S for focus and which touch the three sides of each of the triangles ABC, AEF, BFD, CDE, have their latera-recta equal.

Solution by the PROPOSER.

Reciprocate with respect to S; then we have the three altitudes of an equilateral triangle passing through a point, and the circumscribing circles of the four triangles formed by joining the feet of the perpendiculars equal.

The latera recta of the given conics are then equal.

87. **Proposed by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military Academy, Washington, Miss.**

Given any two straight lines in space, AB, CD, which do not intersect. So construct upon one of the lines as base, a triangle, having its vertex in the other line, such that its perimeter shall be a minimum.

No solution of this problem has been received.

88. **Proposed by FREDERICK R. HONEY, Ph. B., Instructor in Mathematics in Trinity College, New Haven, Conn.**

Prove that the vólume of the frustum of a cone is equal to one-sixth of the altitude multiplied by the sum of the areas of the upper base, the lower base, and four times the area of the section midway between the upper and lower bases.

Solution by FREMONT CRANE, Sand Coulee, Mont.; ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa; G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va., and the PROPOSER.

Let R=radius of the lower base; r=radius of the upper base; ρ=radius

of the base mid way between the upper and lower bases ; and a=altitude of frustum. Then $\rho=\frac{1}{2}(R+r)$. $\therefore 4\rho^2=(R+r)^2$.

Volume of frustum$=\frac{1}{3}\pi a(R^2+r^2+Rr)=\frac{1}{6}\pi a(2R^2+2r^2+2Rr)=\frac{1}{6}\pi a[R^2+r^2+(R+r)^2]=\frac{1}{6}\pi a(R^2+r^2+4\rho^2)$.

The same method applies to frustums of pyramids, and all solids coming under the prismatoid formula as special cases.

Also solved in a more general manner by *B. F. SINE*, *P. S. BERG*, *J. SCHEFFER*, *HARVEY N. DAVIS*, *W. H. DRANE*, and *CHAS. C. CROSS*.

CALCULUS.

68. Proposed by EDWARD DRAKE ROE, JR., A. M., Associate Professor of Mathematics, Oberlin College, Oberlin, Ohio.

If $a^{x^{x^{\cdot^{\cdot^{x}}}}}$ to r steps be denoted by $\overset{r}{a^x}$, and if $y=\overset{r}{x^x}$, prove that

$$D_xy=x^{\overset{r}{x}+\overset{r-1}{x}+\dots+\overset{1}{x}}(\log x)^{r-1}(1+\log x)+\sum_{k=2}^{k=r}x^{\overset{r}{x}+\overset{r-1}{x}+\dots+\overset{k-1}{x}-1}(\log x)^{r-k}.$$

Solution by the PROPOSER.

If $y=f_1(x)^{f_2(x)}$, we obtain, by taking the logarithm of both sides of the equation, and differentiating, the formula

$$D_xy=f_1(x)^{f_2(x)}\log f_1(x)D_xf_2(x)+f_1(x)^{f_2(x)-1}f_2(x)D_xf_1(x).$$

If in this $f_1(x)=x$, $f_2(x)=x^x$, that is if $y=x^{\overset{x}{x}}=\overset{2}{x^x}$, we obtain

$$D_xy=x^{\overset{2}{x}+\overset{1}{x}}\log x(1+\log x)+x^{\overset{2}{x}+\overset{1}{x}-1},$$

and the formula to be proved is true when $r=2$. It is evidently not true for values of $r<2$. Assume that it is true for all other values of r.

Let $y=x^{\overset{r+1}{x}}=x^{\overset{r}{x^x}}$. In the above formula put, $f_1(x)=x$, $f_2(x)=\overset{r}{x^x}$, and we get

$$D_xy=x^{\overset{r+1}{x}}\log xD_x\overset{r}{x^x}+x^{\overset{r+1}{x}-1}\overset{r}{x^x},$$

but by this assumption this is

$$D_xy=x^{\overset{r+1}{x}}\log x[x^{\overset{r}{x}+\overset{r-1}{x}+\dots+\overset{1}{x}}(\log x)^{r-1}(1+\log x)]+x^{\overset{r+1}{x}}\log x\sum_{k=2}^{k=r}x^{\overset{r}{x}+\overset{r-1}{x}+\dots+\overset{k-1}{x}-1}(\log x)^{r-k}$$
$$+x^{\overset{r+1}{x}+\overset{r}{x}-1}$$
$$=x^{\overset{r+1}{x}+\overset{r}{x}+\overset{r-1}{x}+\dots+\overset{1}{x}}(\log x)^{r}(1+\log x)+\sum_{k=2}^{k=r+1}x^{\overset{r+1}{x}+\overset{r}{x}+\dots+\overset{k-1}{x}-1}(\log x)^{r+1-k}.$$

But this expression has the same form with respect to $r+1$, that the assumption had with respect to r, and since the assumption was true for $r=2$, it is also true for all values of r greater than 2, which is what we had to prove.

Erlangen, Bayern, Hauptstrasse 83II, 26 February, 1898.

Also solved by *C. W. M. BLACK*, *W. W. LANDIS*, and *G. B. M. ZERR*.

69. Proposed by **GEORGE LILLEY, Ph. D., LL. D.**, Professor of Mathematics, University of Oregon, Eugene, Oregon.

An elliptic fence encloses a field whose major and minor axes are $2a$ and $2b$, respectively. The ends of a rope, the length of which is equal to the length of the fence, are fastened outside the fence and at the extremities of the major axis. A horse is tethered by means of a ring which slides freely on the rope. Over how much ground can he feed? What is the length of the outside border? Find these values in square feet and feet, true to six decimal places, when the area of the field is one acre and $a=2b$.

Solution by **MELLEN WOODMAN HASKELL, A. M., Ph. C.**, Associate Professor of Mathematics, University of California, Berkeley, Cal.

The outside border will evidently be a curve parallel to the given ellipse, so that the two curves will have common normals. Let ds denote the element of arc on the ellipse, dS the element of arc on the parallel curve lying between the same normals, $d\phi$ the angle between those normals, dA the element of area bounded by those normals, by ds and dS, and p the length of the rope, then

$$dS=ds+pd\phi, \text{ and } dA=pds+\tfrac{1}{2}p^2d\phi.$$

Integrating around the ellipse, since in this case the perimeter of the ellipse $\int ds$ is equal to p, and also the complete integral $\int d\phi$ is evidently 2π, we have

$$S=p(1+2\pi) \text{ and } A=p^2(1+\pi).$$

A simple calculation then gives

$p=806.693$ feet, perimeter of ellipse;
$S=5875.293$ feet, length of outside border;
$A=2695155$ square feet, included area.

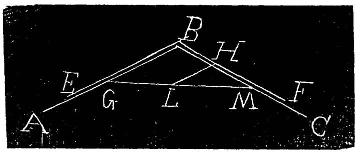
MECHANICS.

57. Proposed by **J. C. NAGLE, M. A., M. C. E.**, Professor of Civil Engineering, Agricultural and Mechanical College of Texas, College Station, Texas.

Over the intersection of two inclined planes slides a cord of uniform mass throughout its length. Find the equation to the path described by its center of gravity.

Solution by the PROPOSER.

Let AB, BC be the inclined planes, EBF the position of the cord at any instant. Let the length of the cord be a; let BC be the x-axis, AB the y-axis. Let $EB=z$. Then the center of gravity of EB will be at G, the mid-point of EB, and of BF at M, the mid-point of BF. Let L be the center of gravity of whole cord. By moments about M,



$GM\times z=LM\times a$, or $GM/LM=a/z$.........(1).

But from similar triangles,

$$\frac{GM}{LM}=\frac{BM}{HM}=\frac{\frac{1}{2}(a-z)}{\frac{1}{2}(a-z)-x}=a/z.$$

Whence $2az=z^2+a^2-2ax$ ....(2).

Also from the figure, $GM/LM=\frac{1}{2}z/y=a/z$, from which $z=\sqrt{(2ay)}$.........(3).

Substitute this value of z in (2), square and arrange terms, and there results,

$$4x^2-8xy+4y^2-4ax-4ay+a^2=0 \text{ ....(4).}$$

Equation (4) is that of a parabola, tangent to the axes at a distance of $\frac{1}{2}a$ from the origin.

59. Proposed by **WILLIAM HOOVER, A. M., Ph. D.**, Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

Find the radius of a sphere of given specific gravity which will rest just immersed in a fluid whose density varies as its depth.

I. Solution by **HENRY HEATON, M. Sc.**, Atlantic, Iowa; **E. L. SHERWOOD, A. M.**, Superintendent of Schools, West Point, Miss., and **W. W. LANDIS, A. M.**, Professor of Mathematics, Dickinson College, Carlisle, Pa.

Put r=radius of sphere, s_1=its specific gravity, s_2=specific gravity of the fluid at the depth of 1 foot, and a=62.5 pounds=weight of a cubic foot of water. The weight of the sphere is $\frac{4}{3}as_1\pi r^3$. The weight of the sphere must equal the weight of the fluid displaced. To find this, put x=the distance of a horizontal section of the sphere below the surface. The weight of an equivalent quantity of the fluid between two sections whose distance apart is dx is $as_2\pi(r^2-x^2)xdx$. Hence the weight of fluid displaced is

$$as_2\pi\int_0^{2r}(r^2-x^2)xdx=as_2\pi r^4=\tfrac{4}{3}as_1\pi r^3. \quad \therefore\ r=s_1/s_2.$$

II. Solution by **ALFRED HUME, C. E., D. Sc.**, Professor of Mathematics, University of Mississippi, University, Miss.; and **J. C. NAGLE, M. A., M. C. E.**, Professor of Civil Engineering, in State Agricultural and Mechanical College, College Station, Texas.

Let the origin of coördinates be taken at the lowest point of the sphere, the axis of x vertical, that of y horizontal. The equation of the section of the sphere made by the xy-plane is $y^2=2Rx-x^2$, R being the sphere's radius.

If ρ be the density of the liquid at a depth of unity, then at a distance x above the origin, the density is $\rho(2R-x)$. The weight of the displaced liquid is

$$\int_0^{2R}\pi y^2dx.\rho(2R-x)g.$$

If c be the density of the sphere, its weight is $\frac{4}{3}\pi R^3cg$.

These being equal

$$\tfrac{4}{3}cR^3=\rho\int(2Rx-x^2)(2R-x)dx,$$

the limits being $2R$ and 0, $=\frac{4}{3}\rho R^4$; $R=c/\rho$.

III. Solution by the **PROPOSER**.

Let δ, δ' be the respective densities of the sphere and of the fluid at a unit depth, and take the upper extremity of the vertical diameter of the sphere for origin, that diameter being in the axis of x.

$\delta' x$=the density of the fluid at depth x, and for equilibrium the mass of sphere immersed equals the mass of fluid displaced.

Then, r being the radius of the sphere,

$$\delta.\tfrac{4}{3}\pi r^3=\int\int\pi y^2 dx.\delta x=\int_0^{2r}\pi\delta' x(2rx-x^2)dx=\delta'\tfrac{4}{3}\pi r^4. \quad \therefore\ r=\delta/\delta'.$$

Also solved by *J. SCHEFFER* and *G. B. M. ZERR*.

60. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

What must be the ratio of the two legs of a uniform and heavy right triangle suspended from the center of the inscribed circle, if this triangle will rest with the shorter leg in a horizontal position ?

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

If a and b be the longer and shorter side about the right angles, and r the radius of the inscribed circle, we should have

$$\tfrac{1}{2}r[a+b+\sqrt{(a^2+b^2)}]=\tfrac{1}{2}ab\ \dots\dots\dots\dots\dots\dots\dots(1),$$

$$\text{or}\quad r=\tfrac{1}{2}a+\tfrac{1}{2}b-\tfrac{1}{2}\sqrt{(a^2+b^2)}\ \dots\dots\dots\dots\dots\dots(2).$$

The triangle is kept in equilibrium by the action of the weights of the sides vertically downwards and the reaction of the point of support.

Taking moments about this fallen point,

$$ra=(\tfrac{1}{2}b-r)\sqrt{(a^2+b^2)}+b(\tfrac{1}{2}b-r)\ \dots\dots\dots\dots\dots\dots(3),$$

the sides being taken proportional to their weights.

r from (2) put into (3) gives, by reduction,

$$3a=4b\ \text{ or }\ a=\tfrac{4}{3}b\ \dots\dots\dots\dots\dots\dots\dots\dots(4).$$

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa, and W.W. LANDIS, A. M., Professor of Mathematics, Dickinson College, Carlisle, Pa.

If a and b represent the legs of the triangle the distance of the center of gravity from the leg a is $\frac{1}{3}b$. The radius of the inscribed circle is

$$r=\tfrac{1}{2}[a+b-\sqrt{(a^2+b^2)}].$$

To secure equilibrium in the desired position r must equal $\frac{1}{3}b$, or

$$\tfrac{1}{3}b=\tfrac{1}{2}[a+b-\sqrt{(a^2+b^2)}].$$

Whence $b=\frac{3}{4}a$, or $b/a=\frac{3}{4}$.

In this triangle the centee of gravity is directly above the center of the inscribed circle. Hence the equilibrium is unstable.

III. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and J. SCHEFFER, A. M., Hagerstown, Md.

In order that AB may be horizontal the point of suspension O and the center of gravity G must be in the same straight line perpendicular to AB.

Let $AB=x$, $BC=mx$, $AC=x\sqrt{(1+m^2)}$. Then $HB=r$=radius of in-circle.

$$2(x-r)+2(mx-r)+2r=x+mx+x\surd(1+m^2).$$

$\therefore\ r=\frac{1}{2}x[1+m-\surd(1+m^2)]$.

$GH=\frac{1}{3}BC=\frac{1}{3}mx,\ KH=\frac{1}{2}AB-HB=\frac{1}{2}x-r$.

$\therefore\ KH=\frac{1}{2}x[\surd(1+m^2)-m)],\ GK=\frac{1}{3}CK=\frac{1}{3}x\surd(\frac{1}{4}+m^2)$.

$\therefore\ KG^2=GH^2+KH^2$, or $\frac{1}{9}x^2(\frac{1}{4}+m^2)=\frac{1}{9}m^2x^2+\frac{1}{4}x^2[\surd(1+m^2)-m]^2$.

$\therefore\ \frac{1}{4}[\surd(1+m^2)-m]^2=\frac{1}{36}$, or $\surd(1+m^2)-m=\frac{1}{3}$. $\therefore\ m=\frac{4}{3}$.

$\therefore\ AB:BC=3:4$, and $HB:AB:BC:AC=1:3:4:5$.

Also solved by *J. C. NAGLE.*

DIOPHANTINE ANALYSIS.

60. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

It is required to find six positive numbers, such that if each be diminished by five-half times the fifth power of their sum the six remainders will be rational fifth powers.

Solution by the PROPOSER.

Let u, v, w, x, y, z, be the six numbers required, and let $u+v+w+x+y+z=s$.

Then $u-\frac{5}{2}s^5=h^5s^5/q^5,\quad v-\frac{5}{2}s^5=k^5s^5/q^5,\quad w-\frac{5}{2}s^5=l^5s^5/q^5$,

$x-\frac{5}{2}s^5=m^5s^5/q^5,\quad y-\frac{5}{2}s^5=n^5s^5/q^5,\quad z-\frac{5}{2}s^5=p^5s^5/q^5$,

Adding these six equations we get

$$s-15s^5=(s^5/q^5)(h^5+k^5+l^5+m^5+n^5+p^5).$$

Let $h^5+k^5+l^5+m^5+n^5+p^5=q^5$. $\therefore\ s=\frac{1}{2}$.

$\therefore\ u=\frac{1}{32}[\frac{5}{2}+(h^5/q^5)],\quad v=\frac{1}{32}[\frac{5}{2}+(k^5/q^5)],\quad w=\frac{1}{32}[\frac{5}{2}+(l^5/q^5)]$,

$x=\frac{1}{32}[\frac{5}{2}+(m^5/q^5)],\quad y=\frac{1}{32}[\frac{5}{2}+(n^5/q^5)],\quad z=\frac{1}{32}[\frac{5}{2}+(p^5/q^5)]$.

Let $h=4$, $k=5$, $l=6$, $m=7$, $n=9$, $p=11$, $q=12$.

$u=\frac{1217}{15552}$, $v=\frac{625205}{7962624}$, $w=\frac{81}{1024}$, $x=\frac{638887}{7962627}$, $y=\frac{2803}{32768}$, $z=\frac{783131}{7962624}$.

Let $h=5$, $k=10$, $l=11$, $m=16$, $n=19$, $p=29$, $q=30$.

$u=\frac{19441}{248832}$, $v=\frac{1217}{15552}$, $w=\frac{60911051}{777600000}$, $x=\frac{3862411}{48600000}$, $y=\frac{63226099}{777600000}$, $z=\frac{81261149}{777600000}$.

AVERAGE AND PROBABILITY.

58. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

From a point on the surface of a circle two lines are drawn to the circumference. Required the average area that may be cut from the circle in this way if the lines are supposed to be drawn at equal angular intervals.

Query I. How does this differ from problem 32?

Query II. Is *sector* the proper word to use for the surface thus cut off?

Query III. Is it absolutely correct to use the word *random* in average problems?

I. Solution by the PROPOSER.

For each pair of lines a second pair may be drawn in opposite directions, dividing the surface of the circle into four portions each of which is included between two of the lines and the circumference. Hence the whole number of surfaces thus cut off may be arranged in sets of four such that the areas of each set shall equal the area of the circle. Hence the average required is $\frac{1}{4}a^2\pi$, where a is the radius of the circle.

Query I. As problem 32 does not describe how the lines are to be drawn to form the "sector" this is a particular case of that problem.

Query II. This query was proposed for information. Some one may be able to give authority for the use of the word in this sense. It is contrary to the usual definition.

Query III. It is the opinion of the writer that the use of the word *random* in average problems is the result of confusion of ideas, and although sometimes convenient is never correct.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let P be the given point. Through P draw the two chords MN, SR dividing the surface of the circle into the four surfaces A, B, C, D.

Then $A+B+C+D=\pi r^2$.

Since P can be taken anywhere on the surface of the circle and the lines MN, SR can make any angle from 0 to π, the average area of A=average area of B=average area of C=average area of D.

$\therefore\ A=B=C=D=\frac{1}{4}\pi r^2$.

After carefully examining problem 32 I am inclined to think the above result the true answer to that problem also.

59. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A circle is rolling along a horizontal straight line. The uniform velocity of the center is v. Find the average velocity of a point of the circumference.

Solution by JOHN M. COLAW, A. M., Monterey, Va.; JOSIAH H. DRUMMOND, LL. D., Portland, Me.; M. E. GRABER, Mt. Vernon, O.; and the PROPOSER.

For the cycloid traced by the point, we have

$$\left.\begin{array}{l} x=a\theta-a\sin\theta \\ y=a-a\cos\theta \end{array}\right\},$$

$$dx=a(1-\cos\theta)d\theta\ ;\ dy=a\sin\theta d\theta.$$

$\therefore\ ds^2=dx^2+dy^2,=2a^2d\theta^2(1-\cos\theta),=2a^2d\theta^2(2\sin^2\frac{1}{2}\theta)$.

$\therefore\ ds=2a\sin\frac{1}{2}\theta d\theta$.

Now $OT=vt=a\theta$. $\therefore\ dt=(a/v)d\theta$.

$\therefore\ ds/dt=2a\sin\frac{1}{2}\theta d\theta\div(a/v)d\theta,=2v\sin\frac{1}{2}\theta$, the variable velocity of P.

$$\therefore \text{ the required average} = \frac{2v\int_0^\pi \sin\frac{1}{2}\theta d\theta}{\int_0^\pi d\theta} = 4v/\pi.$$

Also solved by *G. B. M. ZERR*.

MISCELLANEOUS.

56. Proposed by S. HART WRIGHT, A. M., M. D., Ph. D., Penn Yan, N. Y.

In latitude 40° N.$=\lambda$, when the moon's declination is $5^\circ\ 23'$ N.$=\delta$, and the sun's declination $9^\circ\ 52'$ S.$=-\delta'$, how long after sunset will the cusps of the moon's crescent set synchronously, the moon having recently passed its conjunc tion with the sun ?

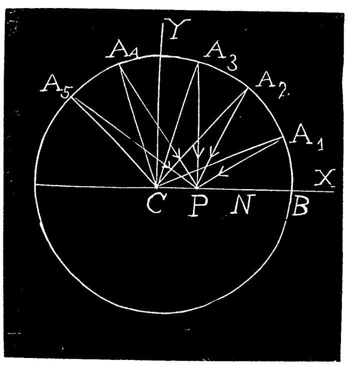
[NOTE. Problem 56 is identical with problem 54, and need not be here reproduced. See January number, pages 27 and 28, for two solutions. EDIROR.]

57. Proposed by GEORGE LILLEY, Ph. D., Professor of Mathematics, University of Oregon, Eugene, Ore.

A particle is placed very near the center of a circle, round the circumference of which n equal repulsive forces are symmetrically arranged; each force varies inversely as the mth power of its distance from the particle. Show that the resultant force is approximately $\dfrac{m_1 n(m-1)}{2r^{m+1}}\times CP$, and tends to the center of the circle, where m_1 is the mass of the particle, CP its distance from the center of the circle, and r the radius of the circle.

I. Solution by the PROPOSER.

Let the particle be at P, and C the center of the circle. Suppose the forces to be at A_1, A_2, ... $CP=x$, and $\angle A_2CB=\theta$. Then $\angle A_1CA_2=\angle A_2CA_3$ $=\ldots\ldots=360^\circ/n=\beta$, say. Draw A_2N at right angles to CB. Consider the force at A_2. Then,



$$X=[m_1/(A_2P)^m]\cos A_2PN=[m_1(r\cos\theta-x)]/(r^2+x^2-2rx\cos\theta)^{\frac{1}{2}(m+1)}.$$

Let $m_1/r^{\frac{1}{2}(m+1)}=M$. Then, since x is small, neglecting terms containing higher powers of x than the first, we have

$$X=M\left[r^{\frac{1}{2}(1-m)}\cos\theta+\frac{\cos 2\theta . x . r^{-\frac{1}{2}(m+1)}}{2}+\tfrac{1}{2}(m-1)xr^{-\frac{1}{2}(m+1)}\right].$$

To obtain the total force on P along X take the sum of n such expressions for all values of θ from $\theta=\alpha$ to $\theta=\alpha+(n-1)\beta$. Hence,

$$\Sigma X = M\Big[\frac{r^{\frac{1}{2}(1+m)}\cos[\alpha+\frac{1}{2}(n-1)\beta]\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}+\frac{\cos[2\alpha+(n-1)\beta]\sin n\beta}{2\sin\beta}xr^{-\frac{1}{2}(m+1)}$$

$$+n(m-1)xr^{-\frac{1}{2}(m+1)}\Big]=\frac{m_1n(m-1)}{2r^{m+1}}x.$$

Similarly, for the force along Y,

$Y=[m_1/(A_2P)^m]\sin A_2PN=Mr\sin\theta(1-2x\cos\theta)^{-\frac{1}{2}(m+1)}=k\sin\theta+k_1x\sin2\theta$, where $k=Mr^{\frac{1}{2}(1-m)}$ and $k_1=M(m-1)r^{-\frac{1}{2}(m+1)}$.

For the sum of n such expressions, $\Sigma Y=0$. Hence, the resultant is

$$[(\Sigma X)^2+(\Sigma Y)^2]^{\frac{1}{2}}=\frac{m_1n(m-1)}{2r^{m+1}}\times CP.$$

Since the forces are equal and symmetrically arranged in the circumference, their resultant will act *towards* the center of the circle.

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

It is assumed that the forces are placed at equal intervals along the circumference, that the repulsive action is proportional to the mass acted on, and that each force acts with unit intensity upon a unit mass at a unit's distance.

Because of the symmetrical arrangement the resultant force will act in the direction PC, C being the center of the circle and P the position of the particle whose mass is m_1.

Let the line connecting the forces with C make angles of α, 2α, 3α, etc., with the radius through P, so that $n\alpha=2\pi$.

The distance of the first force from P is $\sqrt{(r^2-2r.CP.\cos\alpha+CP^2)}$, the cosine of the angle which its direction makes with CP is

$$(r\cos\alpha-CP)/\sqrt{(r^2-2r.CP.\cos\alpha+CP^2)},$$

and the force itself is

$$m_1/(r^2-2r.CP.\cos\alpha+CP^2)^{\frac{1}{2}m}.$$

Writing d for CP, the component of this force along PC is

$$m_1(r\cos\alpha-d)/(r^2-2r\ d\ \cos\alpha+d^2)^{\frac{1}{2}(m+1)}.$$

Putting this in the form

$$m_1(r\cos\alpha-d)(r^2-2r\ d\ \cos\alpha+d^2)^{-\frac{1}{2}(m+1)},$$

expanding by the binomial theorem, multiplying, and neglecting the third and higher powers of d, this becomes,

$$m_1r^{-m-1}\{-d+[r-\tfrac{3}{2}(m+1)r^{-1}d^2]\cos\alpha+(m+1)d\cos^2\alpha$$
$$+\tfrac{1}{2}[m+1)(m+3)]r^{-1}d^2\cos^3\alpha\}.$$

The sum of all such components is

$$m_1 r^{-m-1}\{-nd+[r-\tfrac{3}{2}(m+1)r^{-1}d^2](\cos\alpha+\cos 2a+\cos 3\alpha+\ .....\cos n\alpha)$$
$$+(m+1)d(\cos^2\alpha+\cos^2 2\alpha+\cos^2 3\alpha+......\cos n\alpha)$$
$$+\tfrac{1}{2}[(m+1)(m+3)]r^{-1}d^2(\cos^3\alpha+\cos^3 2\alpha+\cos^3 3\alpha+......\cos^3 n\alpha)\}.$$

By trigonometry,

$$\cos\alpha+\cos 2\alpha+..\ ...\cos n\alpha=\frac{\cos\frac{1}{2}(n+1)\alpha.\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha};$$

$$\cos^2\alpha+\cos^2 2\alpha+.....\ \cos^2 n\alpha=\tfrac{1}{2}\{n+\frac{\cos(n+1)\alpha.\sin n\alpha}{\sin\alpha}\};$$

$$\cos^3\alpha+\cos^3 2\alpha+......\cos^3 n\alpha=\frac{\cos\frac{1}{2}(n+3)\alpha.\sin\frac{3}{2}n\alpha}{4\sin\frac{3}{2}\alpha}+\frac{3\cos\frac{1}{2}(n+1)\alpha.\sin\frac{1}{2}n\alpha}{4\sin\frac{1}{2}\alpha}.$$

The value of these series when $n\alpha=2\pi$ are 0, $n/2$, and 0, respectively.

Hence the expression for the approximate value of the resultant force reduces to

$$m_1 r^{-m-1}[-nd+(\tfrac{1}{2}n)(m+1)d],\ \text{or}\ \frac{m_1 n(m-1)}{2r^{m+1}}d.$$

Also solved by *G. B. M. ZERR.*

PROBLEMS FOR SOLUTION.

ARITHMETIC.

95. **Proposed by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military College, Washington, Miss.**

Solve by arithmetic, if possible.

A man sold a house for \$7500 and gained a certain per cent. on the cost. If the cost had been $16\frac{2}{3}\%$ less, his gain would have been 25% greater. Find the cost of the house.

96. **Proposed by RAYMOND SMITH, Tiffin, Ohio.**

How many acres in a square field whose diagonal is 10 rods longer than the side ?

*** Solutions of these problems should be sent to B. F. Finkel, not later than April 10.

ALGEBRA.

85. **Proposed by J. M. COLAW, A. M., Monterey, Va.**

Sum the infinite series

$$\frac{1}{1^2.2^2.3^2.4^2.5^2}+\frac{1}{2^2.3^2.4^2.5^2.6^2}+\frac{1}{3^2.4^2.5^2.6^2.7^2}+,\ \text{etc.}$$

86. **Proposed by J. MARCUS BOORMAN, Consultative Mechanician, Counselor at Law, Inventor, Etc., Woodmere, Long Island, N. Y.**

Solve $x^2+yz=16$ .....(A); $y^2+xz=17$......(B); $z^2+xy=22$......(C), for all the roots.

[This is Col. Titus' problem—see "Maseres' Tracts," pages 188-276—and is solved by Dr. Wallis in 51 pages, and by Mr. Frend in 38 pages, 8vo., but by the writer in 1 or 2 pages, 4to., or less. J. M. B.]

87. **Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.**

A starts to travel around a circular island at a given point and travels at the rate of 5 miles in 4 hours. One half hour after A, B starts from a point directly opposite from A and travels in an opposite direction at the rate of 4 miles in 3 hours. One hour afterwards C starts from the same point as A and travels in an opposite direction to A at the rate of 3 miles in 2 hours. One half hour afterwards D starts from the same point as B and travels in an opposite direction to B at the rate of 2 miles in 1 hour. Required the size of the island, and when they will all be together, and how far each will have traveled at the accomplishment of this event.

*** Solutions of these problems should be sent to J. M. Colaw, not later than June 10.

GEOMETRY.

94. **Proposed by EDMOND FISH, Hillsboro, Ill.**

A tower $AB=a$, is surmounted by a flag pole $BC=b$. A point D is so taken in a line perpendicular to the foot of the tower that angle BDC is a maximum. Prove that AD is a mean proportional between AC and AB.

95. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.**

At each point of a parabola is described the rectangular hyperbola of four-pointic contact; prove that the locus of the center of the hyperbola is an equal parabola.

96. **Proposed by W. F. BRADBURRY, A. M., Head Master, Cambridge Latin School, Cambridge, Mass.**

Isosceles triangles are constructed externally on the three sides of a triangle as bases, with the angles at the bases each 30°. The triangle formed by joining the remote vertices (the 120° vertices) of these isosceles triangles is equilateral. [Geometric--not Trigonometric—solution.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than June 10.

CALCULUS.

75. **Proposed by O. W. ANTHONY, M. Sc., Instructor in Mathematics in Boys' High School, New York City.**

Solve the differential equation

$$\frac{d^2y}{dx^2}+n^2y=\frac{6}{x^2}y.$$

76. **Proposed by E. B. ESCOTT, Cambridge, Mass.**

Solve the partial differential equation

$$q^2r+4pqs+p^2t+p^2q^2(rt-s^2)=a^2.$$

[*Forsyth's Differential Equations*, page 376.]

77. **Proposed by T. E. COLE, Columbus, Ohio.**

Derive the equation of a point in the pedal of a bicycle as the wheel rolls along on a plane.

*** Solutions of these problems should be sent to J. M. Colaw, not later than June 10.

MECHANICS.

66. Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College. Springfield, Mo.

A conical stick of timber, length a, radius of base r, and density δ, is depressed, apex downward, in a liquid, density δ', so that the base is just level with the liquid. If left free to rise, required the greatest altitude to which it will ascend.

67. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Find the horizontal and vertical components of the moon's "disturbing force" for any point on the earth's surface making an angle φ with the line joining the center of the earth to the center of the moon.

*** Solutions of these problems should be sent to B. F. Finkel, not later than June 10.

AVERAGE AND PROBABILITY.

63. Proposed by COL. CLARKE.

Three points are taken at random, one on each of the three faces of a tetrahedron; what is the chance that a plane passing through them cuts the fourth edge?

[From *Williamson's Integral Calculus*, page 410.]

64. Proposed by Rev. W. A. WHITWORTH, A. M.

O is a given point within a triangle; P is a random point within the same. The line through O and P is produced so as to divide the triangle into a trapezium and a triangle. Find the average area of this triangle. [From the *Educational Times*, London, Eng.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than June 10.

BOOKS AND PERIODICALS.

Theoretical and Practical Graphics. By Frederick N. Willson, C. E., A. M., Professor in the School of Science, Princeton University. (Author's Edition) 1897. 4to. Pages viii+264+Appendix.

This is a most attractive work, not only conquering graphics entire, but containing much more of highest geometric interest, including a fairly complete course on higher plane curves.

The part of the subject where Church so long held supremacy in America with his Descriptive Geometry justly appreciated for its elegance, is paralleled by Professor Willson in his Chapter I and Chapters IX—XII, 117 pages in all, including 219 figures in the text, where he not only covers with equal conciseness and elegance the matter of Church's 138 pages of text and 21 pages of illustrations (102 figures), but in addition has treated many new and important matters, such as the Conoid of Pluecker (articles 333, 356, 477) a favor-

ite surface of Sir Robert Ball, applied in his Theory of Screws, which itself may be looked upon as in part an application of non-Euclidean geometry, also the Cylindroid of Frézier (§§ 333, 360, 489), the corne de vache (§ 361, 475-6), and some special helicoids (§ 480-4), and has also covered the Third Angle (or 'shop') Method of employing descriptive geometry, and given a very full treatment of development (§ 405-20). The mathematical surfaces are beautifully illustrated.

The general plan of the book, while providing a comprehensive graphical training in the form of a progressive course, admits of specialization, of shorter courses, with noticeable flexibility. In fact, eight subgroupings are indicated for independent courses. Comparison with the special treatises scrupulously cited shows the extent of matter on all topics usually treated to be surprisingly great. Professor Willson has a gift for condensing without loss of clearness. With this power, he does well to restate for convenient reference many of the fundamental definitions which he presumes already in some form previously mastered, for example the definition of the trigonometric functions on page 31.

But I still prefer the definition in the note on page 121, "a straight line is the line which is completely determined by two points," to the author's second thought given in the preface, "the line that is completely determined by any two of its points." The spheric space of non-Euclidean geometry, though movable as a whole in itself, is such that two geodetic lines in it always cut in two points. Of course no spherical trigonometry is employed in the author's solution of the problems of trihedrals, pure a graphic process, as it should be. We are glad to find as an appendix the brief but very weighty paper on Trochoids which was so highly and justly praised when presented to the American Association for the Advancement of Science. We cannot forbear to dwell upon the superb illustrations, which make the book a portfolio of art. The author is particularly happy in deciding conflicts of nomenclature, as where he refuses to follow Javary (§ 508) in calling the geodesic on a cone a conical helix.

The author has been extraordinarily painstaking in the proof-reading, and the book is practically free from error. A few trifles have been noticed—

Page 165, § 433, first line, for "prism" read "cylinder."
Page 171, § 442, first line, for "axes" read "bases."
Page 37, sixth line from below, for 90° read 9°.
Page 67, § 194, seventh line, for ϕ read θ.

The slip on page 55, § 166, in stating the brachistochrone and tautochrone properties of the cycloid is so evidently a reference to a reversed or inverted figure inadvertently omitted that it also is trivial.

As to the briefest hint of contents by chapters: I. Definitions, classification. II. Free-hand sketching. III. Draughtsman's outfit. IV. Use of instruments. V. Higher plane curves. VI. Conventional representation. VII. Lettering. The treatment of lettering is particularly full, and sixty-four alphabets are given. VIII. Copying processes. IX. Descriptive geometry of Monge. X. Projections, intersections, development of surfaces, with applications to elbow joints, blast pipes, arch construction, &c. XI. Trihedrals. XII. Projection of sphere. Here the now disused orthographic projection is somewhat condensed, but the stereographic, which is used, is far more complete than in Church. XIII. Shades and shadows. XIV. Perspective. XV. and XVI. Isometric and Clinographic projection, with applications; also, crystals in oblique projection. XVII. Bridge details, toothed gearing, &c. Out of a host of beautiful figures, we may mention 92 as particularly efficient in teaching homology or *complete plane perspective.*

It is a particular pleasure to welcome this book, because it is on just the lines where English and American mathematics has hitherto been sterile.

Even now, the tremendous, the fundamental importance of von Staudt's geometry of position, the pure projective geometry, both for science and philosophy, is realized by few. For example, in the Bolyai type of non-Euclidean geometry, not only is the straight line infinite, but also it has two distinct points at infinity; it is never closed, even by points

at infinity. Writing in 1835, even the superhuman penetration of Lobachévski attributed this essential openness to the straight in itself. In the introduction to his "New Elements of Geometry" he says: "I consider it unnecessary to analyse in detail other assumptions too artificial or arbitrary. Only one of them still deserves some attention; namely, the passing over of the circle into a straight line. Moreover, here the fault is visible from the beginning in the violation of continuity, when a curve, which does not cease to be closed, however great it may be, must change immediately into the most infinite straight line, since in this way it loses an essential characteristic.

In this regard the imaginary geometry (the non-Euclidean geometry) fills out the interval much better. When in it we increase a circle all whose diameters come together at a point; finally, we so attain to a line such that its normals continually approach, although they no longer can cut one another. This characteristic does not pertain to the straight, but to the curve, which, in my paper "On the Foundations of Geometry," I have called *circle-limit.* Of course, it was not until in the next decade (1847) that von Staudt published his immortal "Geometrie der Lage, but long afterward Helmholtz suffers still more seriously for lack of the pure projective geometry, treating the projective questions which necessarily came up in his extended optical researches, sometimes by means and methods of his own make; sometimes only by general reasonings.

Again in *Mind* (1876) Helmholtz misses thus a fundamental difference. He says, p. 315: "It is in fact possible to imagine conditions for bodies apparently solid such that the measurements in Euclid's space become what they would be in spherical or pseudo-spherical space. * * * Think of the image of the world in a convex mirror. * * * Now, Beltrami's representation of pseudo-spherical space in a sphere of Euclid's space is quite similar, except that the background is not a plane, as in the convex mirror, but the surface of a sphere, and that the proportion in which the images as they approach the spherical surface contact, has a different mathematical expression."

But in reality these differences are so fundamental as to make all the difference between Euclidean and non-Euclidean; for the changed measure for distance in the mirror world is still Euclidean, parabolic, using an imaginary conic in the plane background as "absolute" in Cayley's sense. Thus Helmholtz repeated, reproduced the old, but false, theorem that in space of positive curvature two geodetic lines, if they in general cut, must necessarily cut in *two* points. He never attained the conception of single elliptic space, the type-form, but speaks only of "spherical space of three dimensions."

It is to be hoped that Professor Willson's book may hasten the day in America when courses in descriptive geometry and pure projective geometry, no longer confined to science schools, may be available in every college, and when there may be a more adequate realization of the power of spatial imaging as an instrument in scientific research.

Austin, Texas. DR. GEORGE BRUCE HALSTED.

New Psychology. By John P. Gordy, Ph. D., LL. D., Head of the Pedagogical Department of the Ohio State University. 8vo. Cloth, 402 pages. Price, $1.25. New York: Hinds & Noble.

This is the best elementary Psychology that has yet appeared. The plan of this work is the same as that of Dr. Gordy's "Lessons in Psychology," a book published in 1891 and, probably, read by a larger number of teachers than any similar work on the subject. The New Psychology is entirely rewritten, and the subject is treated in a simple, clear and philosophical manner. It should be in the hands of every teacher who desires to acquire a good working knowledge of the development and activity of the human mind. B. F. F.

Mechanical Drawing. By J. C. Tracy, C. E., Instructor in the Sheffield Scientific School of Yale University. 1898. New York: Harper & Brothers.

This book affords an excellent introductory course. Its aim is to prepare the student

for a more extended course in any one of the special lines of drafting. It is comprehensive enough for use in schools and colleges, and, at the same time, is admirably suited to the needs of the student who must study the subject with little, if any, help from a teacher. The book closes with a valuable chapter on Perspective by E. H. Lockwood, M. E., instructor in the Sheffield Scientific School. J. M. C.

Supplemento al Periodico di Mathematica. Diretto dal Dott, Giulio Lazzeri.

Étude sur le Triangle et sur Certains Points de Geometrographie. Par M. E. Lemoine, Ancien élève de l'École Politechnique, á Paris. Pamphlet 24 pages.

Reprint from the proceedings of the Edinburgh Mathematical Society, Vol. XIII.

Elementi di Geometria. By G. Lazzeri e A Bassani Professori nella R. Accademia Navale, Livorno, Italia. Large 8vo, paper covers, 380 pages.

This work is divided into five books. Book I contains five chapters. Chapter I treats of lines and planes; chapter II, of segments, angles and dihedrals; chapter III, first notions on circles and spheres; chapter IV, parallel lines and parallel planes; chapter V, lines and planes perpendicular. Book II contains four chapters. Chapter I treats of polygons; chapter II, of solid angles; chapter III, of polyhedria; chapter IV, of distances. Book III contains four chapters. Chapter I treats of the relation between lines, planes and spheres; chapter II, of relations of polygons with circles and of polyhedria with spheres; chapter III, of geometry of the sphere; chapter IV, of surfaces and solids of revolution. Book IV contains seven chapters. Chapter I treats of the general theory of equivalence; chapter II, of the equivalence of polygons and of polyhedric surfaces; chapter III, of the equivalence of spherical polygons and spherical pyramids; chapter IV, of equivalence of prisms; chapter V, of the extent of limits; chapter VI, of the equivalence of polyhedra; chapter VII, of equivalence of circles and the three round bodies—sphere, cylinder and cone. Book V contains five chapters. Chapter I treats of the theory of proportion; chapter II, of homology and similarity; chapter III, of measurement; chapter IV, of the application of algebra to geometry.

From the above outline of the contents of this work, it is clear to be seen that the authors have not followed the old stereotyped order of presenting the subject of geometry, but have treated it in a wholly original manner. The figures of solid geometry are very artistic, representing, as near as it is possible for diagrams to do, the real spatial figure. The book contains 312 figures. At the end of each book there is a long list of theorems for original work, making a total of 1,067. B. F. F.

Mélanges Sur la Géométrie du Triangle. Par M. E. Lemoine, Ancien élève de l'École Politecnique, à Paris.

This paper was presented at the meeting of the French Association for the Advancement of Science, August 8, 1895. In this paper, Mr. Lemoine has derived many interesting properties of the triangle, and established a number of theorems in reference to the triangle and its relation to conic sections. In the Geometry of the Triagle, as studied from the modern point of view, Mr. Lemoine is the recognized leader and authority. B. F. F.

Yale Entrance Examinations in Mathematics. Compiled by Richard Mather, Ph. B. 1898. New Haven, Conn.: Boardman School Press.

This book contains the Yale entrance examinations in methematics from 1884 to 1898, and furnishes a very interesting and useful collection. The book will prove especially acceptable to preparatory and other schools where students are being fitted for college.

J. M. C.

Application de la Geometrographic a l'Examen de Diverses Solutions d'un même Problème. Par M. E. Lemoine, Ancien élève de l'École Politecnique, à Paris. · Extract du *Bulletin de la Société Mathematique de France.* Pamphlet, 19 pages.

Questions Relatives a la Géométrie du Triangle, a la Geometrographie et a la Transformation Continue. Par M. E. Lemoine, Ancien éléve de l'École Politecnique, á Paris. Pamphlet, 32 pages.

A paper presented to the French Associatioa for the Advancement of Science, at its meeting April 3, 1896.

Integral Calculus. By Daniel Alexander Murray, Ph. D., Instructor in Mathematics in Cornell University. 288 pages. 1898. New York and Chicago: American Book Company.

This book is one of the Cornell mathematical series. While written primarily for use at Cornell, this text is admirably suited for any one beginning this study. The first two chapters, treating, respectively, of "integration, a process of summation" and "integration, the inverse of differentiation," are more extended than is usual in elementary works, and will prove an important aid to the student in gaining a clear idea of what the Integral Calculus is, and of the uses to which it may be applied. The subject matter is presented in a simple manner, but there has been no sacrifice of rigor of treatment. Chapters of more than ordinary interest are those on Integral Curves and on Ordinary Differential Equations. Many practical problems and illustrative examples are found throughout the book, while the appendix contains important additional matter. In short, the matter, contents and details of treatment characterize a well-constructed text-book, which is worthy the attention of teachers. J. M. C.

The American Journal of Mathematics, January, 1898, has the following leading papers: The Motion of a Solid in Infinite Liquid under no Forces, by A. G. Greenhill; Surfaces of Rotation with Constant Measure of Curvature and their Representation on the Hyperbolic (Cayley's) Plane, by Geo. F. Metzler; Sur les méthodes d'Approximations successives dans la Théorie des Equations différentielles, par Emile Picard. There is a frontispiece portrait of G. Darboux.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year, in advance. Single number, 25 cents. The Review of Reviews Co., New York.

In history-making times like these a truthful record of passing events becomes an imperative need. The daily newspaper is ephemeral and not easily preserved for reference. The *American Monthly Review of Reviews* has all the value of the newspaper, besides distinctive merits of its own. As an epitome of current history it is complete, compact, terse, impartial, absolutely reliable, and judiciously edited. As a piece of journalistic history-writing, what could be more brilliant or fascinating than the May number of this publication, with its story of the Spanish-American war crisis? Merely as a souvenir of this past eventful month the *Review* has a certain unique fitness.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Irvington-on-the-Hudson, New York.

One of the most popular literary magazines published in America.

The American Journal of Mathematics, for April, has three interesting papers, as follows: On the Focal Surfaces of the Congruences of Tangents to a given Surface, by A. Pell; Displacements Depending on One, Two and Three Parameters in a Space of Four Dimensions, by Thomas Craig; and, Further Researches in the Theory of Quintic Equations, by Emory McClintock.

The following periodicals have been received since the last acknowledgement: *Journal de Mathématiques Élémentaires*, (1er Mars, 1898); *The American Journal of Mathematics*, (January, 1898); *L' Intermédiaire des Mathématiciens*, (Fevrier 1898); *Miscellaneous Notes and Queries*, (Nov.-Dec., 1897); *Bulletin of the American Mathematical Society*, (February, 1898); *The Kansas University Quarterly*, (January, 1898); *The Monist*, (January, 1898); *The Literary Digest*, (January to March 19, 1898); *The Ohio Teacher*, (March, 1898); *The Educational Times*, (March 1, 1898); *American Journal of Mathematics*, (April, 1898); *L' Intermédiaire des Mathématiciens*, (Mars 1898); *Journal de Mathématiques Élémentaires*, (15 Mars 1898); *The Monist*, (April, 1898); *Bulletin of the American Mathematical Society*, (April, 1898); *The Mathematical Gazette*, (February, 1898); *The Educational Times*, (April, 1898).

NOTES.

In a letter from Dr. Alexander Macfarlane, among other things he says:

The following extract from Dr. Laisant's book "La Mathématique: Philosophic Ensciguement" may encourage the editors of the MONTHLY.

Parmi toutes les nations du monde, il u' en est assurément par une plus intérsessante que les Etats-Uuis d' Amerique, au point de vue du développment rapide qu a pris ce pays, des progrés prodigieux de son industrie, de l' énergieet de l' initiative dont il a fait preuve. Mais tant d' activité obligée, imposeé à un pays neuf par les conditions de son existence, se concileait mal avec des recherches pureuant théoriques, avec la poursuite sentimentale de la verité pure. Aussi le nombre des mathématiciens Americains a-t-il été longtemps des plus reduits. Il y a environ vingt-cing aus, mon ami regretté J. Houël l' un des savants les instruits de sou temps, et dont l' érudition mathématique était immense, m' aître tenait dans une de ses lettres de la situation mathématique des diverses nations et arrivait a' cette phrase typique dont je n' ai jamais perdu le souvenir. "Quant aux Etats-Unis, ils importent juste la quantité de science pure que est necessaire à leur industrie."

C' était très' exact alors. Dupuis cette époque, au savant anglais, Sylvester, mort il y a peu de temps, fut appelé a professer a Baltimore; il y fonda un important journal mathématique, et provoqua une sorte de révolution intellectuelle à ce point de vue, en quelques années seulement. Aujour d' hui, la *Societé Mathematique Americaine* de New York comprend à peu près le même nombre

de membres que la *Societé Mathématique de France.* La Mathématique sons toutes ses formes et dans toutes ses parties, est professée dans une foule de publications et cultiveé par des savants qui ne le cé dent en rien à leurs confréres d' Europe. Elle n' est plus au object d' importation emprunté a l' ancien monde ; c' est devenu un article essential de la production nationale, et cette production augmente chaque join comme importance et comme quantité. Ce phénomène s' est accompli, je le repete, en un trés petit nombre d' années et il est assez curieux pour valoir la peine d' une indication.

Malgré ce développment extraordinaire, et peut-être à cause de ce développment l' industrie Americaine n' a rien perdu de son activité, bien au contraire, elle prend à tâche et parfois avec une sorte de fièvre, de transporter les résultats de la science pure dans le domaine des applications dès qu' elle les juge utilisables ; et c' est par centaines que l' on pourrait compter les publications Americaines s' occupant chaque jour, sous une forme ou sous une autre, de Mathématique appliquée.

Believing this to be of interest to our readers also, we have published the extract in full.

ERRATA.

Page 27, line 2 from bottom, for "meridian" read *vertical circle.*

Page 43, line 11, for "quadrate" read (*bi-*)*quadrate*; line 18, after "15" insert "Add, etc., $x^2-2xy+y^2=-25$. $\therefore \pm(x-y)=5i$ at once."

Page 44, line 12, after (II), insert "as any figures for (XIII)......(XV) or (M) will show"; line 22, for "or" read *and*, and insert (VI) before "above"; line 30, omit "(" before $20\frac{1}{2}$; line 33, for "$x=$" read $x_{||}=$; line 38, for "in" read *into.*

Page 86, line 7, for "$2^{n-3}2(2^2-1)s+t$" read $2^{n-3}[2(2^2-1)s+t]$; line 8, for "$2^{n-4}2(2^3-1)s+t$" read $2^{n-4}[2(2^3-1)s+t]$; line 11, for "$b=x_2=2^{n-1}t$" read $b=x_2=2^{n-2}t$.

Page 89, line 28, after "$8r/\pi$" insert *log* before expression in parenthesis.

Page 90, line 3, omit "to"; line 15, read $\rho=r[1-(4/\pi)\theta]$.

Page 93, line 5 from bottom, for "power" read *powers*, and in next line, for "sum" read *sums.*

Page 95, line 13, for "$\frac{1}{2}P^3$" read $\frac{1}{2}P^5$, and for "S_5" read S_6; line 14, for "....1/42" read ....+1/42 ; in problem 83, read $y^2+x=60$....(2).

Page 96, in problem 73, read $\int_0^{\frac{1}{4}\pi}\log(1+\tan x)dx$; problems 64 and 65 in Diophantine Analysis, should be 66 and 67.

Page 96, problem 92, for $AB\times BC : DC\times AD=BD : AC$, read $AB\times BC+DC\times AD : AB\times AD+BC\times CD=BD : AC$.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. MAY, 1898. No. 5.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from March Number.]

PROPOSITION XXXIV, *in which is investigated a certain curve arising from the hypothesis of acute angle.* [*An equidistantial of a straight has its chords between it and the straight.*]

Let the straight CD join equal perpendiculars AC, BD standing upon any straight AB. Then AB, CD being bisected in the points M and H (Fig. 42.), MH is joined perpendicular (by Proposition II) to each. Again in this hypothesis the angles at the join CD are supposed acute. Therefore in the quadrilateral $AMHC$ (by Corollary I after Proposition III) MH will be less than AC. Hence now, if in MH produced

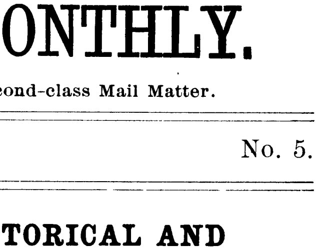


Fig. 42.

MK be taken equal to AC, the points C, K, D pertain to the curve here investigated. Then the angles at the join CK will be themselves acute (by Proposition VII).

Therefore the join LX, which bisects, and therefore (by Proposition II), is at right angles to AM, CK, will be likewise (by Corollary I after Proposition III) less than AC. Wherefore, if in LX produced we assume LF equal to AC or MK, the point F also will pertain to this curve. Further, joining CF, and

FK we find likewise two other points pertaining to the same curve. And so on for ever.

But what I say for finding points between the points C and K, the same also holds good uniformly for finding points between the points K and D. Obviously the curve CKD, arising from the hypothesis of acute angle, is the line joining the extremities of all equal perpendiculars erected upon the same base toward the same part, which assuredly can come under the name ordinates. It is, I add, a line of such sort, that on account of the hypothesis of acute angle, from which it arises, it always is concave toward the parts of the opposite base AB. Quod quidem hoc loco declarandum, ac demonstrandum a nobis erat.

PROPOSITION XXXV. *If from any point L of the base AB the ordinate LF is drawn to this curve CKD: I say the straight NFX perpendicular to LF must on both sides fall wholly toward the convex parts of this curve, and therefore it will be tangent to this curve.*

Proof. For if possible, let a certain point X (Fig. 43.) of NFX fall within the cavity of this curve. Let fall from the point X to the base AB the perpendicular XP, which prolonged through X meets the curve in a certain point R. Now thus. In the quadrilateral $LFXP$ the angle at the point X will be neither right nor obtuse: else (Proposition V and Proposition VI) would be destroyed the present hypothesis of acute angle.

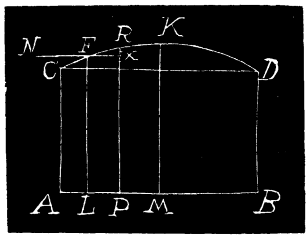


Fig. 43.

Therefore the aforesaid angle will be acute. Wherefore (from Corollary I after Proposition III) PX and so much more PR will be greater than LF. But this is absurd (from the preceding) against the nature of this curve.

So NF produced must fall wholly toward the convex parts, and so it will be tangent to this curve. Quod erat demonstrandum.

[To be Continued.]

SUMMATION OF SERIES.

By G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

$$\frac{\sin h\theta}{\theta}=\left(1+\frac{\theta^2}{\pi^2}\right)\left(1+\frac{\theta^2}{2^2\pi^2}\right)\left(1+\frac{\theta^2}{3^2\pi^2}\right)\left(1+\frac{\theta^2}{4^2\pi^2}\right)\ldots\ldots$$

$$=1+\frac{\theta^2}{3\,!}+\frac{\theta^4}{5\,!}+\frac{\theta^6}{7\,!}+\frac{\theta^8}{9\,!}+\ldots\ldots(A).$$

$$\therefore \log\left(1+\frac{\theta^2}{\pi^2}\right)+\log\left(1+\frac{\theta^2}{2^2\pi^2}\right)+\log\left(1+\frac{\theta^2}{3^2\pi^2}\right)+\log\left(1+\frac{\theta^2}{4^2\pi^2}\right)\ldots\ldots$$

$$=\log\left[1+\left(\frac{\theta^2}{3!}+\frac{\theta^4}{5!}+\frac{\theta^6}{7!}+\frac{\theta^8}{9!}+\cdots\cdots\right)\right].$$

$$\therefore \frac{\theta^2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\cdots\right)$$

$$-\frac{\theta^4}{2\pi^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\cdots\cdots\right)$$

$$+\frac{\theta^6}{3\pi^6}\left(\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\cdots\cdots\right)$$

$$-\frac{\theta^8}{4\pi^8}\left(\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\cdots\cdots\right)\cdots\cdots$$

$$=\frac{\theta^2}{6}-\frac{\theta^4}{180}+\frac{\theta^6}{2835}-\frac{\theta^8}{37800}+\cdots\cdots$$

$$\therefore \frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\cdots\right)=\frac{1}{6},$$

$$\text{and}\ \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\cdots=\frac{\pi^2}{6}\ \cdots\cdots\cdots\ (1),$$

$$-\frac{1}{2\pi^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\cdots\cdots\right)=-\frac{1}{180}$$

$$\text{and}\ \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\cdots\cdots=\frac{\pi^4}{90}\ \cdots\cdots\cdots\ (2),$$

$$\frac{1}{3\pi^6}\left(\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\cdots\cdots\right)=\frac{1}{2835}$$

$$\text{and}\ \frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\cdots\cdots=\frac{\pi^6}{945}\ \cdots\cdots\cdots\ (3),$$

$$-\frac{1}{4\pi^8}\left(\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\cdots\cdots\right)=-\frac{1}{37800}$$

$$\text{and}\ \frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\cdots\cdots=\frac{\pi^8}{9450}\ \cdots\cdots\cdots\ (4),$$

$$\cosh\theta=\left(1+\frac{4\theta^2}{\pi^2}\right)\left(1+\frac{4\theta^2}{3^2\pi^2}\right)\left(1+\frac{4\theta^2}{5^2\pi^2}\right)\left(1+\frac{4\theta^2}{7^2\pi^2}\right)\cdots\cdots$$

$$=1+\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+\frac{\theta^6}{6!}+\frac{\theta^8}{8!}+\ \cdots\cdots\cdots\cdots\cdots\cdots\ (B).$$

$$\therefore \log\left(1+\frac{4\theta^2}{\pi^2}\right)+\log\left(1+\frac{4\theta^2}{3^2\pi^2}\right)+\log\left(1+\frac{4\theta^2}{5^2\pi^2}\right)+\ \ldots\ldots$$

$$=\log\left[1+\left(\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+\frac{\theta^6}{6!}+\frac{\theta^8}{8!}+\ \ldots\ldots\right)\right].$$

$$\therefore \frac{4\theta^2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ \ldots\ldots\right)-\frac{16\theta^4}{2\pi^4}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ \ldots\ldots\right)$$

$$+\frac{64\theta^6}{3\pi^6}\left(\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\ \ldots\ldots\right)-\frac{256\theta^8}{4\pi^8}\left(\frac{1}{1^8}+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\ \ldots\ldots\right)$$

$$\ldots\ldots=\frac{\theta^2}{2}-\frac{\theta^4}{12}+\frac{\theta^6}{45}-\frac{17\theta^8}{2520}+\ \ldots\ldots$$

$$\therefore \frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ \ldots\ldots\right)=\tfrac{1}{2}$$

$$\text{and } \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ \ldots\ldots=\frac{\pi^2}{8}\ \ldots\ldots\ldots\ldots\ (5),$$

$$-\frac{16}{2\pi^4}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ \ldots\ldots\right)=-\tfrac{1}{12}$$

$$\text{and } \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ \ldots\ldots=\frac{\pi^4}{96}\ \ldots\ldots\ldots\ (6),$$

$$\frac{64}{3\pi^6}\left(\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\ \ldots\ldots\right)=\tfrac{1}{45}$$

$$\text{and } \frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\ \ldots\ldots=\frac{\pi^6}{960}\ \ldots\ldots\ldots\ (7),$$

$$-\frac{256}{4\pi^8}\left(\frac{1}{1^8}+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\ \ldots\ldots\right)$$

$$\text{and } \frac{1}{1^8}+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\ \ldots\ldots=\frac{17\pi^8}{161280}\ \ldots\ldots\ (8).$$

(1)−(5) gives $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ \ldots\ldots=\frac{\pi^2}{24}\ \ldots\ldots\ldots\ldots\ldots\ (9),$

(2)−(6) gives $\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\ \ldots\ldots=\frac{\pi^4}{1440}\ \ldots\ldots\ldots\ldots\ldots\ (10),$

(3)−(7) gives $\frac{1}{2^6}+\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\ \ldots\ldots=\frac{\pi^6}{60480}\ \ldots\ldots\ldots\ldots\ (11),$

(4)−(8) gives $\frac{1}{2^8}+\frac{1}{4^8}+\frac{1}{6^8}+\frac{1}{8^8}+\ldots\ldots=\frac{\pi^8}{2419200}$ .... (12),

(5)−(9) gives $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\ldots\ldots=\frac{\pi^2}{12}$ ..... (13),

(6)−(10) gives $\frac{1}{1^4}-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\frac{1}{6^4}+\ldots\ldots=\frac{7\pi^4}{720}$ ...... (14).

(7)−(11) gives $\frac{1}{1^6}-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\ldots\ldots=\frac{31\pi^6}{30240}$ .... (15).

(8)−(12) gives $\frac{1}{1^8}-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\frac{1}{5^8}-\frac{1}{6^8}+\ldots\ldots=\frac{11\pi^8}{172800}$ ...... (16),

$\frac{1}{2}$ of (1) gives $\frac{1}{1.2}+\frac{1}{2.4}+\frac{1}{3.6}+\frac{1}{4.8}+\ldots\ldots=\frac{\pi^2}{12}=(13)$ .... (17),

$\frac{1}{2}$ of (5) gives $\frac{1}{1.2}+\frac{1}{3.6}+\frac{1}{5.10}+\frac{1}{7.14}+\ldots\ldots=\frac{\pi^2}{16}$ .... (18),

$\frac{1}{3}$ of (1) gives $\frac{1}{1.3}+\frac{1}{2.6}+\frac{1}{3.9}+\frac{1}{4.12}+\ldots\ldots=\frac{\pi^2}{18}$ .... (19),

$\frac{1}{3}$ of (5) gives $\frac{1}{1.3}+\frac{1}{3.9}+\frac{1}{5.15}+\frac{1}{7.21}+\ldots\ldots=\frac{\pi^2}{24}=(9)$ .... (20),

$\frac{1}{4}$ of (1) gives $\frac{1}{1.4}+\frac{1}{2.8}+\frac{1}{3.12}+\frac{1}{4.16}+\ldots\ldots=\frac{\pi^2}{24}=(9)=(20)$ .. (21).

$\frac{1}{4}$ of (5) gives $\frac{1}{1.4}+\frac{1}{3.12}+\frac{1}{5.20}+\frac{1}{7.28}+\ldots\ldots=\frac{\pi^2}{32}$ .... (22).

(17)−(18) gives $\frac{1}{2.4}+\frac{1}{4.8}+\frac{1}{6.12}+\frac{1}{8.16}+\ldots\ldots=\frac{\pi^2}{48}$ .... (23),

(19)−(20) gives $\frac{1}{2.6}+\frac{1}{4.12}+\frac{1}{6.18}+\frac{1}{8.24}+\ldots\ldots=\frac{\pi^2}{72}$ .... (24),

(21)−(22) gives $\frac{1}{2.8}+\frac{1}{4.16}+\frac{1}{6.24}+\frac{1}{8.32}+\ldots\ldots=\frac{\pi^2}{96}$ .... (25),

(1)−(2) gives $\frac{1}{1^2}-\frac{1}{2^4}+\frac{1}{3^2}-\frac{1}{3^4}+\frac{1}{4^2}-\frac{1}{4^4}+\frac{1}{5^2}-\frac{1}{5^4}+\ldots\ldots=\frac{\pi^2}{6}\left(1-\frac{\pi^2}{15}\right)$

$\therefore \frac{3}{2^4}+\frac{8}{3^4}+\frac{15}{4^4}+\frac{24}{5^4}+\ldots\ldots=\frac{\pi^2}{6}\left(1-\frac{\pi^2}{15}\right)$ .... (26).

(2)—(3) gives $\frac{3}{2^6}+\frac{8}{3^6}+\frac{15}{4^6}+\frac{24}{5^6}+\ldots\ldots=\frac{\pi^4}{45}\left(\frac{1}{2}-\frac{\pi^2}{21}\right)$ ....(27),

(3)—(4) gives $\frac{3}{2^8}+\frac{8}{3^8}+\frac{15}{4^8}+\frac{24}{5^8}+\ldots\ldots=\frac{\pi^6}{945}\left(1-\frac{\pi^2}{10}\right)$ ....(28),

(5)—(6) gives $\frac{8}{3^4}+\frac{24}{5^4}+\frac{48}{7^4}+\frac{80}{9^4}+\ldots\ldots=\frac{\pi^2}{8}\left(1-\frac{\pi^2}{12}\right)$ ....(29),

(6)—(7) gives $\frac{8}{3^6}+\frac{24}{5^6}+\frac{48}{7^6}+\frac{80}{9^6}+\ldots\ldots=\frac{\pi^4}{96}\left(1-\frac{\pi^2}{10}\right)$ ....(30),

(7)—(8) gives $\frac{8}{3^8}+\frac{24}{5^8}+\frac{48}{7^8}+\frac{80}{9^8}+\ldots\ldots=\frac{\pi^6}{960}\left(1-\frac{17\pi^2}{168}\right)$ ....(31),

(9)—(10) gives so also (26)—(29)

$$\frac{3}{2^4}+\frac{15}{4^4}+\frac{35}{6^4}+\frac{63}{8^4}+\ldots\ldots=\frac{\pi^4}{24}\left(1-\frac{\pi^2}{60}\right)\ldots\ldots(32),$$

(10)—(11) gives so also (27)—(30)

$$\frac{3}{2^6}+\frac{15}{4^6}+\frac{35}{6^6}+\frac{63}{8^4}+\ldots\ldots=\frac{\pi^4}{1440}\left(1-\frac{\pi^2}{42}\right)\ldots\ldots(33),$$

(11)—(12) gives so also (28)—(31)

$$\frac{3}{2^8}+\frac{15}{4^8}+\frac{35}{6^8}+\frac{63}{8^8}+\ldots\ldots=\frac{\pi^6}{60480}\left(1-\frac{\pi^2}{40}\right)\ldots\ldots(34),$$

$\frac{1}{8}$ of (29) gives $\frac{1}{3^4}+\frac{3}{5^4}+\frac{6}{7^4}+\frac{10}{9^4}+\ldots\ldots=\frac{\pi^2}{64}\left(1-\frac{\pi^2}{12}\right)$ ....(35),

$\frac{1}{8}$ of (30) gives $\frac{1}{3^6}+\frac{3}{5^6}+\frac{6}{7^6}+\frac{10}{9^6}+\ldots\ldots=\frac{\pi^2}{768}\left(1-\frac{\pi^2}{10}\right)$ ....(36),

$\frac{1}{8}$ of (31) gives $\frac{1}{3^8}+\frac{3}{5^8}+\frac{6}{7^8}+\frac{10}{9^8}+\ldots\ldots=\frac{\pi^6}{7680}\left(1-\frac{17\pi^2}{168}\right)$ .....(37).

Squaring (1) gives $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots$

$$+\frac{2}{1^2.2^2}+\frac{2}{1^2.3^2}+\frac{2}{2^2.3^2}+\ldots\ldots=\frac{\pi^4}{36}.$$

$$\therefore\frac{1}{1^2.2^2}+\frac{1}{1^2.3^2}+\frac{1}{1^2.4^2}+\frac{1}{2^2.3^2}+\frac{1}{2^2.4^2}+\frac{1}{3^2.4^2}+\ldots\ldots$$

$$=\tfrac{1}{2}\left(\frac{\pi^4}{36}-\frac{\pi^4}{90}\right)=\frac{\pi^4}{120}=\frac{\pi^4}{5\,!}\;\ldots\ldots\ldots\ldots\ldots\ldots (38).$$

Squaring (2) gives with (4)

$$\frac{1}{1^4.2^4}+\frac{1}{1^4.3^4}+\frac{1}{1^4.4^4}+\frac{1}{2^4.3^4}+\frac{1}{2^4.4^4}+\frac{1}{3^4.4^4}+\ldots$$

$$=\tfrac{1}{2}\left(\frac{\pi^8}{8100}-\frac{\pi^8}{9450}\right)=\frac{\pi^8}{113400}\;\ldots\ldots\ldots\ldots\ldots (39).$$

Squaring (5) gives with (6)

$$\frac{1}{1^2.3^2}+\frac{1}{1^2.5^2}+\frac{1}{1^2.7^2}+\frac{1}{3^2.5^2}+\frac{1}{3^2.7^2}+\frac{1}{5^2.7^2}+\ldots$$

$$=\tfrac{1}{2}\left(\frac{\pi^4}{64}-\frac{\pi^4}{96}\right)=\frac{\pi^4}{384}=\frac{\pi^2}{4^2(4!)}\;\ldots\ldots\ldots\ldots\ldots (40).$$

Squaring (6) gives with (8)

$$\frac{1}{1^4.3^4}+\frac{1}{1^4.5^4}+\frac{1}{1^4.7^4}+\frac{1}{3^4.5^4}+\frac{1}{3^4.7^4}+\frac{1}{5^4.7^4}+\ldots$$

$$=\tfrac{1}{2}\left(\frac{\pi^2}{9216}-\frac{17\pi^8}{161280}\right)=\frac{\pi^8}{645120}\;\ldots\ldots\ldots\ldots\ldots\ldots (41).$$

Squaring (9) gives with (10)

$$\frac{1}{2^2.4^2}+\frac{1}{2^2.6^2}+\frac{1}{2^2.8^2}+\frac{1}{4^2.6^2}+\frac{1}{4^2.8^2}+\frac{1}{6^2.8^8}+\ldots\ldots$$

$$=\tfrac{1}{2}\left(\frac{\pi^4}{576}-\frac{\pi^2}{1440}\right)=\frac{\pi^4}{1920}\;\ldots\ldots\ldots\ldots\ldots\ldots (42).$$

Squaring (10) gives with (12)

$$\frac{1}{2^4.4^4}+\frac{1}{2^4.6^4}+\frac{1}{2^4.8^4}+\frac{1}{4^4.6^4}+\frac{1}{4^4.8^4}+\frac{1}{6^4.8^4}+\ldots\ldots$$

$$=\tfrac{1}{2}\left(\frac{\pi^8}{2073600}-\frac{\pi^8}{241920}\right)=\frac{\pi^8}{29030400}\;\ldots\ldots\ldots (43).$$

$$1/\left[\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots\right]=\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\ldots\right)$$

$$\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\ldots\right)\left(1+\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\ldots\right)\ldots .$$

$$=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots=\tfrac{1}{6}\pi^2 \quad ..(44),$$

$$1/\left[\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\ldots\ldots\right]$$

$$=\left(1+\frac{1}{2^4}+\frac{1}{2^8}+\frac{1}{2^{12}}+\ldots\right)\left(1+\frac{1}{3^4}+\frac{1}{3^8}+\frac{1}{3^{12}}+\ldots\right)$$

$$\left(1+\frac{1}{5^4}+\frac{1}{5^8}+\frac{1}{5^{12}}+\ldots\right)\ldots$$

$$=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots\ldots=\frac{\pi^4}{90} \quad \ldots\ldots\ldots (45).$$

Similarly,

$$1/\left[\left(1-\frac{1}{2^6}\right)\left(1-\frac{1}{3^6}\right)\left(1-\frac{1}{5^6}\right)\left(1-\frac{1}{7^6}\right)\ldots\ldots\right]=\frac{\pi^6}{945}\ldots\ldots\ldots (46).$$

$$1/\left[\left(1-\frac{1}{2^8}\right)\left(1-\frac{1}{3^8}\right)\left(1-\frac{1}{5^8}\right)\left(1-\frac{1}{7^8}\right)\ldots\ldots\right]=\frac{\pi^8}{9450}\ldots\ldots\ldots (47).$$

Hence,

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots\ldots=\frac{6}{\pi^2}\ldots\ldots\ldots (48).$$

$$\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\ldots\ldots=\frac{90}{\pi^4}\ldots\ldots\ldots (49).$$

$$\left(1-\frac{1}{2^6}\right)\left(1-\frac{1}{3^6}\right)\left(1-\frac{1}{5^6}\right)\left(1-\frac{1}{7^6}\right)\ldots\ldots=\frac{945}{\pi^6}\ldots\ldots\ldots (50).$$

$$\left(1-\frac{1}{2^8}\right)\left(1-\frac{1}{3^8}\right)\left(1-\frac{1}{5^8}\right)\left(1-\frac{1}{7^8}\right)\ldots\ldots=\frac{9450}{\pi^8}\ldots\ldots\ldots (51).$$

$$\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\left(1+\frac{1}{7^2}\right)\ldots\ldots$$

$$=\frac{\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\ldots\ldots}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots\ldots}=\frac{\frac{90}{\pi^4}}{\frac{6}{\pi^2}}=\frac{15}{\pi^2}\quad\ldots\ldots\ldots (52).$$

$$\left(1+\frac{1}{2^4}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{5^4}\right)\left(1+\frac{1}{7^4}\right)\ldots .$$

$$=\frac{\left(1-\frac{1}{2^8}\right)\left(1-\frac{1}{3^8}\right)\left(1-\frac{1}{5^8}\right)\left(1-\frac{1}{7^8}\right)\cdots\cdots}{\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\cdots\cdots}=\frac{\frac{9450}{\pi^8}}{\frac{90}{\pi^4}}=\frac{105}{\pi^4}\ \cdots\cdots\cdots (53).$$

$$\left(1+\frac{1}{2^2}+\frac{1}{2^4}\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}\right)\left(1+\frac{1}{5^2}+\frac{1}{5^4}\right)\left(1+\frac{1}{7^2}+\frac{1}{7^4}\right)\cdots\cdots$$

$$=\frac{\left(1-\frac{1}{2^6}\right)\left(1-\frac{1}{3^6}\right)\left(1-\frac{1}{5^6}\right)\left(1-\frac{1}{7^6}\right)\cdots\cdots}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\cdots\cdots}=\frac{\frac{945}{\pi^6}}{\frac{6}{\pi^2}}=\frac{315}{2\pi^4}\ \cdots\cdots\cdots (54).$$

From (A) by inspection,

$$\frac{1}{1^2.2^2.3^2}+\frac{1}{1^2.2^2.4^2}+\frac{1}{1^2.3^2.4^2}+\frac{1}{2^2.3^2.4^2}+\cdots\cdots=\frac{\pi^6}{7\,!}\cdots\cdots\cdots\cdots\cdots (55).$$

$$\frac{1}{1^2.2^2.3^2.4^2}+\frac{1}{1^2.2^2.3^2.5^2}+\frac{1}{1^2.3^2.4^2.5^2}+\cdots\cdots=\frac{\pi^8}{9\,!}\cdots\cdots\cdots\cdots\cdots (56).$$

Generally $\dfrac{1}{1^2.2^2\ldots(n-1)^2}+\dfrac{1}{1^2.3^2\ldots n^2}+\dfrac{1}{1^2.2^2.4^2\ldots n^2}+\cdots\cdots=\dfrac{\pi^{2n}}{(2n+1)!}\cdots (57).$

When $\theta=\pi$, $\dfrac{1}{\pi}\sinh\pi=\left(1+\dfrac{1}{1^2}\right)\left(1+\dfrac{1}{2^2}\right)\left(1+\dfrac{1}{3^2}\right)\left(1+\dfrac{1}{4^2}\right)\left(1+\dfrac{1}{5^2}\right)\cdots (58).$

$$=1+\frac{\pi^2}{3\,!}+\frac{\pi^4}{5\,!}+\frac{\pi^6}{7\,!}+\frac{\pi^8}{9\,!}+\ldots(59)=1+(1)+(38)+(55)+(56)+\ldots.(60).$$

From (B) by inspection,

$$\frac{1}{1^2.3^2.5^2}+\frac{1}{1^2.3^2.7^2}+\frac{1}{3^2.5^2.7^2}+\cdots\cdots=\frac{\pi^6}{4^3(6\,!)}\cdots\cdots\cdots\cdots\cdots (61).$$

$$\frac{1}{1^2.3^2.5^2.7^2}+\frac{1}{1^2.3^2.5^2.9^2}+\frac{1}{3^2.5^2.7^2.9^2}+\cdots\cdots=\frac{\pi^8}{4^4(8\,!)}\cdots\cdots\cdots\cdots\cdots (62).$$

Generally $\dfrac{1}{1^2.3^2 \text{ to } n \text{ factors}}+\dfrac{1}{1^2.5^2 \text{ to } n \text{ factors}}+\cdots\cdots=\dfrac{\pi^{2n}}{4^n(2n\,!)}\cdots\cdots (63).$

When $\theta=\pi$, $\cosh\pi=\left(1+\dfrac{4}{1^2}\right)\left(1+\dfrac{4}{3^2}\right)\left(1+\dfrac{4}{5^2}\right)\left(1+\dfrac{4}{7^2}\right)\cdots\cdots\cdots (64)$

$$=1+\frac{\pi^2}{2\,!}+\frac{\pi^4}{4\,!}+\frac{\pi^6}{6\,!}+\frac{\pi^8}{8\,!}+.(65)=1+4(5)+4^2(40)+4^3(61)+4^4(62)+.(66).$$

$2\cosh\pi=e^{\pi}+e^{-\pi}=2(1+2^2)[1+(\frac{2}{3})^2][1+(\frac{2}{5})^2][1+(\frac{2}{7})^2]\ \cdots\cdots\cdots\cdots (67).$

$2\sinh\pi=e^{\pi}-e^{-\pi}=4\pi[1+(\frac{1}{2})^2][1+(\frac{1}{3})^2][1+(\frac{1}{4})^2][1+(\frac{1}{5})^2]\ \cdots\cdots\cdots\cdots (68).$

The number of series summed above is deemed sufficient for illustration, although many more could be summed from the above relations.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

90. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find the greatest number of inch balls that can be placed in a box 10 inches square and 5 inches deep.

Solution by MARTIN SPINX, Wilmington, Ohio, and the PROPOSER.

Using as a base a side of the box, we can place on this base, by square arrangement 50 balls. But by placing 5 balls in the first row, 4 in the second, and 5 in the third, and so on, we can place 50 balls in the first layer, there being 6 rows of 5 balls, and 5 rows of 4 balls. These eleven rows will leave .339 inches between the eleventh row and the end of the box. By placing the second layer of balls in the trihedral spaces of the first layer, the two layers will occupy a space 1 inch$+\sqrt{\{1^2-[\frac{2}{3}(\frac{1}{2}\sqrt{3})]^2\}}$ inches,$=1.8165$ inches, high.

Since the centers of the first row in the second layer are .289 inches in advance of the centers of the first row in the first layer, it follows that eleven rows can be put in the second layer, there being .339 inches$-$.289 inches, or .05 inches more room than is needed. But the second layer contains 49 balls,—the six odd rows containing 4 each and the five even rows 5 each. In this we can place in the box twelve layers,—50 balls in each of the odd numbered layers, and 49 in each of the even numbered layers. This makes a total of 594 balls.

Also solved, with different results, by *G. B. M. ZERR*, *CHAS. C. CROSS*, and *FREMONT CRANE*.

90. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

A owes $6000 which is drawing 6% interest. He wishes to pay off the debt in six equal annual payments, the first to be due in one year. The whole portion of the claim unpaid at the ênd of each year to be accounted as principal, and to draw interest to the time of the next payment. Required the amount of each payment, so the six annual payments will discharge the obligation, interest and all.

I. Proposed by P. S. BERG, Superintendent of Schools, Larimore, N. D.; J. A. MOORE. Professor of Mathematics. Millsaps College, Millsaps, Miss.; M. E. GRABER, Mt. Eaton, O.; MARTIN SPINX, Wilmington, O., M. A. GRUBER, A. M., War Department, Washington, D. C.

Let x=annual payment.

Put a=debt, r=rate of interest, and n=number of annual payments.

We then have the general formula,

$$\frac{(100+r)\left\{\begin{array}{c}\dfrac{\dots\dots(100+r)\left(\dfrac{(100+r)a}{100}-x\right)}{100}-x\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \text{to } n\text{th payment}\end{array}\right\}}{100}-x=0,$$

which reduces to

$$a(100+r)^n=x[100(100+r)^{n-1}+100^2(100+r)^{n-2}+\ \dots\ +100^{n-1}(100+r)+100^n].$$

Now, substituting 6 for r, 6 for n, and 6000 for a, we obtain

$$6000\times 1.06^6=x(1.06^5+1.06^4+1.06^3+1.06^2+1.06+1).$$

$\therefore\ x=8511.114673536\div 6.9753185376=1220.1757+$.

II. Solution by F. R. HONEY, Ph. B., New Haven, Conn.

Let x=amount of each annual payment.

$\therefore\ 6000+.06\times 6000-x=6360-x$=amount left after first payment.

And $6360-x+.06(6360-x)-x=6741.6-2.06x$=amount left after second payment.

Similarly, $7146.096-3.1836x$=amount left after third payment.

Similarly, $7574.86176-4.374616x$=amount left after fourth payment.

Similarly, $8029.3534656-5.63709296x$=amount left after fifth payment.

Similarly, $8511.114673536-6.9753185376x$=amount after sixth payment.

Then $8511.114673536-6.9753185376x=0$.

$\therefore\ 6.9753185376x=8511.114673536$.

$\therefore\ x=\$1220.176$=annual payment.

III. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.; W. H. DRANE, A. M., Professor of Mathematics, Jefferson Military College, Washington, Miss.

The portions of the principal paid each year are as

$1,\ 1/1.06,\ 1/(1.06)^2,\ 1/(1.06)^3,\ 1/(1.06)^4,\ 1/(1.06)^5$; or as $(1.06)^5, (1.06)^4, (1.06)^3, (1.06)^2, 1.06, 1$; or as 1.338226, 1.262477, 1.191016, 1.1236, 1.06, 1.

$1.338226+1.262477+1.191016+1.1236+1.06+1=6.976319$.

$(1/6.975319)\times\$6000=\$860.1757$.

$\$860.1756+\$6000\times .06=\$860.1758+\$360=\$1220.1757$, first payment.

$\$6000-\$860.1757=\$5139.8243$, amount still unpaid.

$(1.06/6.975319)\times\$6000=\$911.7862$.

$\$911.7862+\$5139.8243\times .06=\$911.7862+\$308.3895=\$1220.1757$, second payment.

$\$5139.8243-\$911.7862=\$4228.0381$, amount still unpaid.

$[(1.06)^2/6.975319]\times\$6000=\$966.4934$.

$\$966.4934+\$4228.0381\times.06=\$966.4934+\$253.6823=\$1220.1757$, third payment.

$\$4228.0381-\$966.4934=\$3261.5447$, amount still unpaid.

$[(1.06)^3/6.975319]\times\$6000=\$1024.4830$.

$\$1024.4830+\$3261.5447\times.06=\$1024.4830+\$195.6927=\$1220.1757$, 4th payment.

$\$3261.5447-\$1024.4830=\$2237.0617$, amount still unpaid.

$[(1.06)^4/6.975319]\times\$6000=\$1085.9520$.

$\$1085.9520+\$2237.0617\times.06=\$1085.9520+\$134.2237=\$1220.1757$, fifth payment.

$\$2237.0617-\$1085.9520=\$1151.1097$, amount still unpaid.

$[(1.06)^5/6.975319]\times\$6000=\$1151.1091$.

$\$1151.1091+\$1151.1097\times.06=\$1151.1091+\$69.0666=\$1220.1757$, sixth payment.

$\$1151.1097-\$1151.1091=\$0.0006$, unpaid still.

The above is all the work necessary for determining each equal payment, and at the same time working out the problem in full.

A short method by algebra is, $p=\dfrac{6000\times.06(1.06)^6}{(1.06)^6-1}=\$1220.176$.

91. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pa.

$1000.00. Cleveland, Ohio, May 26, 1893.

Two years after date I promise to pay John Davis, or order, one thousand dollars, for value received, interest six per cent. payable annually. J. M. LEWIS.

Indorsements: December 14, 1895, $560.56; May 11, 1896, $10.02; June 14, 1897, $545.06.

Find, by the United States' Rule, the amount due August 2, 1897.

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.; WALTER HUGH DRANE, A. M., Jefferson Military College, Washington, Miss.; and MARTIN SPINX, Wilmington, Ohio.

Principal,		$1000.00
Interest to December 14, 1895, 2 years, 6 months, 18 days,		153.00
Amount,		$1153.00
First payment December 14, 1895,		560.56
New principal,		$ 592.44
Interest to June 14, 1897, 1 year, 6 months,		53.3196
Amount,		$ 645.7596
Second payment, May 11, 1896,	10.02	
Third payment, June 14, 1897,	545.06	555.08
New principal,		$ 90.6796
Interest to August 1, 1897, 1 month, 18 days,		.7254
Amount or balance due,		$91.4050

Daniel G. Dorrance gets as a result $100.80. He computed interest on the annual payments. Mr. Drane, in a second solution does the same, but he gets a result of $100.138.

REMARK. Mr. Drummond raises the question as to whether Mr. J. F. Travis's solution of problem 87, Arithmetic, is strictly an arithmetical solution. To my mind it is strictly an algebraical solution. A pure arithmetical solution of a problem would involve only the operations of addition, subtraction, multiplication, division, involution, and evolution, without the use of equations. A solution in which the result sought is represented by some character, and then this character operated upon until certain conditions of the problem are fulfilled, which conditions are then stated in the form of an equation from which the numerical value of the character is to be determined, is an algebraic solution. It is immaterial what sort of a character is used, whether it be $(\frac{2}{2})$, $\frac{3}{3}$, x, ϕ, or any other character. However, the solution referred to is a very good one, and by the use of such solutions students in arithmetic are given, unconsciously to themselves, a most excellent preparation for the study of algebra. The mathematician is often called upon to solve problèms in a certain way. When a problem is proposed and the restriction put upon it, viz., that it be solved by arithmetic, or algebra, or geometry, the problem often becomes impossible. From such unfortunate restrictions, has arisen the idea of the insolvability of the three famous problems of geometry, viz., the Trisection of an Angle, the Duplication of the Cube, and the Quadrature of the Circle. These problems are each easily solved if the solutions are not restricted to the use of the straight edge and compass only. But with these restrictions they are absolutely unsolvable.

There are many problems whose solutions cannot be effected when restricted in the way previously mentioned, but those referred to above are the only ones that have become famous.

ALGEBRA.

81. II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics, Wesleyan Academy, Wilbraham, Mass.

[See problem and solution I, in April number, page 105.] The proposition cannot be proved unless r is integral and positive, as can be shown by substitution of numerical values.

Consider the only two fractions in whose denominators any factor as (a_1-a_2) appears, putting them in the form

$$(a_1{}^r)/[(a_1-a_2)(a_1-a_3)(a_1-a_4)\ \ldots\ldots(a_1-a_n)]-(a_2{}^r)$$
$$/[(a_1-a_2)(a_2-a_3)(a_2-a_4)\ \ldots\ldots(a_2-a_n)]=(a_1{}^r)$$
$$/[(a_1-a_2)(a_1{}^{n-2}P_1a_1{}^{n-3}+P_2a_1{}^{n-4}-\ldots\ldots\pm P_{n-2})]$$
$$-(a_2{}^r)/[(a_1-a_2)(a_2{}^{n-2}P_1a_2{}^{n-3}+P_2a^{n-4}-\ \ldots\ldots\pm P_{n-2})],$$

where P_k=the sum of the products of a_3, a_4, $\ldots\ldots a_n$ taken k at a time.

Combining, we have

$$[a_1^r(a_2^{n-2}-P_1a_2^{n-3}+P_2a_2^{n-4}-\ldots\ldots\pm P_{n-2})$$
$$-a_2^r(a_1^{n-2}-P_1a_1^{n-3}+P_2a_1^{n-4}-\ldots\ldots P_{n-2})]$$
$$/[(a_1-a_2)(a_1^{n-2}-P_1a^{n-3}+P_2a_1^{n-4}-\ldots\ldots\pm P_{n-2})$$
$$(a_2^{n-2}-P_1a_2^{n-3}+P_2a_2^{n-4}-\ldots\ldots\pm P_{n-2})] \ldots\ldots\ldots\ldots(1).$$

Put the numerator of (1) in the form

$$(a_1^ra_2^{n-2}-a^ra_1^{n-2})-P_1(a_1^ra_2^{n-3}-a_2^ra_1^{n-3})+\ldots\ldots\pm P_{n-2}(a_1^r-a_2^r)$$
$$=a_1^{n-2}a_2^{n-2}(a_1^{r-n+2}-a_2^{r-n+2})-P_1a_1^{n-3}a_2^{n-3}(a_1^{r-n+3}-a_2^{r-n+3})$$
$$+\ldots\ldots\pm P_{n-2}(a_1^r-a_2^r)\ldots\ldots(2).$$

If n is not greater than $r+2$ each group of (2) and consequently the whole expression is divisible by (a_1-a_2). If $n>r+2$, let $n=r+s$; then change (2) to the form

$$a_1^ra_2^r[(a_2^{n-2-r}-a_1^{n-2-r})-P_1(a_2^{n-3-r}-a_1^{n-3-r})+\ldots\ldots$$
$$\pm P_{s-2}(a_2^{n-s-r}-a_1^{n-s-r})]\mp[P_{s-1}a_1^{n-s-1}a_2^{n-s-1}(a_1^{r-n+s+1}-a_2^{r-n+s+1})$$
$$-P_sa_1^{n-s-2}a_2^{n-s-2}(a_1^{r-n+s+2}-a_2^{r-n+s+2})$$
$$+\ldots\ldots\pm P_{n-2}(a_1^r-a_2^r)]\ldots\ldots(3),$$

the term in the second group of (3) being the same as the corresponding term of (2). Each group in (3) is also divisible by (a_1-a_2). Accordingly in all cases (1) can be reduced to a form in which (a_1-a_2) is not a factor of the denominator, and as the two fractions forming (1) are the only ones that contain (a_1-a_2) in their denominators, the original expression need not contain (a_1-a_2) in its denominator; that is, (a_1-a_2) will divide into the numerator formed by adding the fractions as they stand.

In like manner we prove that any other factor (a_2-a_3), etc., will divide into the numerator, or the numerator will be divisible by the entire lowest common denominator.

Now if $r<n-1$ each fraction, and consequently the sum of all, will have a numerator of lower degree in a_1, a_2, a_3, etc., than the denominator. But as the numerator is divisible by the denominator, this is possible only when the numerator equals zero.

If $r=n-1$, numerator and denominator will have same degree, and the quotient can be only a numerical factor. Now in the numerator a_1^r has for its coefficient the product of all factors not containing a_1, which same coefficient it has in the expansion of the denominator. Therefore the quotient must equal 1.

If $r=n$ the numerator is of a degree 1 higher than the denominator and the quotient must be of the first degree. In the numerator the coefficient of a_1^r is the same as the coefficient of $a_1^{n-1}(=a_1^{r-1})$ in denominator, these being the highest powers of a_1 in each. Then one term of the quotient must be a_1. In

like manner we show that a_2, a_3, ...a_n must all be true of quotient, and as the expression is symmetrical with respect to these, the value must be

$$a_1+a_2+a_3+\ldots\ldots+a_n.$$

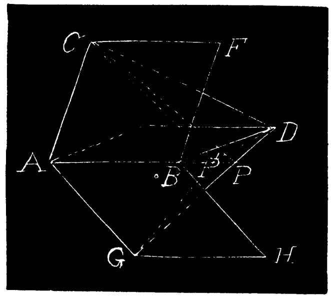
GEOMETRY.

87. Proposed by **WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military Academy, Washington, Miss.**

Given any two straight lines in space, AB, CD, which do not intersect. So construct upon one of the lines as base, a triangle, having its vertex in the other line, such that its perimeter shall be a minimum.

I. Solution by the PROPOSER.

Let AB and CD be the given straight lines. Pass planes through the line AB and the points C and D. In the plane of $ABDE$, inclined the same way and making the same angle with the line AB as $ABFC$, construct a parallelogram equal to $CABF$. Draw DG intersecting AB produced in P. Join PC. Then PCD is the required triangle.



PROOF. Take any other point P' in the line AB. Join $P'D$, $P'C$, and $P'G$. Triangle $P'CA$=triangle $P'GA$ and triangle PCA=triangle PGA. Two sides and included angle being equal in each case. $\therefore$ $P'C=PG$ and $PC=PG$.

Now $P'D+P'G>PD+PG$.

$\therefore$ $P'D+P'C>PD+PC$. Q. E. D.

By passing planes through CD and the points A and B, by a similar construction we may construct a minimum-perimeter triangle upon AB as base with its vertex in CD.

Also solved by *F. R. HONEY*.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let $x+b(a-x)=cy$ be the equation to EF with AB and AY as axes, and AB, EF the given lines, $AB=a$. Then if $\cot\theta+b\cot\varphi=c$ the vertex C will move on EF. Let $AC=r$, $CB=s$.

$$\text{Then } r+s=\frac{a(\sin\theta+\sin\varphi)}{\sin(\theta+\varphi)}=\text{minimum}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

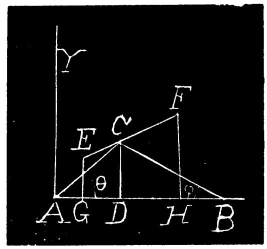
$$\cot\theta+b\cot\varphi=c\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1), $\dfrac{d\theta}{d\varphi}=-\dfrac{\sin\theta}{\sin\varphi}$, from (2), $\dfrac{d\theta}{d\varphi}=-\dfrac{b\sin^2\theta}{\sin^2\varphi}$.

$\therefore$ $\sin\varphi=b\sin\theta$, this in (2) gives

$$\cot\theta=\frac{b^2+c^2-1}{2c},\ \cot\varphi=\frac{c^2-b^2+1}{2bc},\ \cot\theta+\cot\varphi=\frac{AD+DB}{DC}=a/DC,$$

$$\therefore\ DC=\frac{2abc}{b^3-b^2-b+bc^2+c^2+1}.$$



DC must be $>EG$ and $<HF$.

Let $AG=m$, $EG=n$, $AH=h$, $HF=k$.

Then $b=\dfrac{hn-km}{ak+hn-an-km}$, $c=\dfrac{a(h-m)}{ak+hn-an-km}$,

b must be less than unity or EF may intersect AB.

Let $m=n=1$, $h=a=10$, $k=4$. $\therefore\ b=\frac{1}{6}$, $c=\frac{5}{2}$, $DC=\frac{36}{35}$.

$\cot\varphi=\frac{131}{15}$, $\cot\theta=\frac{19}{18}$. $\therefore\ \theta=43°\ 27'\ 6''$, $\varphi=6°\ 31'\ 56''$.

$AD=DC\cot\theta=\frac{38}{35}$.

89. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Describe a circle tangent to three given circles. [From *Chauvenet's Geometry*, page 318, ex. 213.

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.; and F. R. HONEY, Ph. B., Instructor in Mathematics in Trinity College, New Haven, Conn.

In figure 2 let L, M, N be the circles radii a, b, c.

With M as center and radius $b-a$ describe a circle, also with N as center and radius $c-a$ describe a circle. Draw a circle through L tangent to the circles last described at T, S then the center of this circle is the center of one of the tangent circles. Similarly we can find seven other tangent circles.

Of the eight circles one is tangent to the three circles externally, one is tangent internally, three are tangent to two externally and one internally, and three are tangent to two internally and one externally.

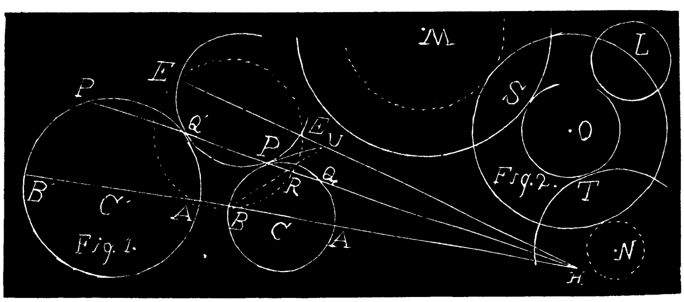


Fig. 1. Fig. 2.

In figure 1, to find a circle passing through a point E and tangent to two circles C, C'. Let H be the point where the external common tangent meets $C'C$ produced. Through $A'BE$ describe a circle cutting EH again in E'. Draw BR meeting HE in U, and draw UP tangent to C, then the circle through PEE' is the circle required.

Two tangents can be drawn from U.

If we had used the point where the internal common tangent cuts CC' we would have determined two other circles, four in all, satisfying the condition.

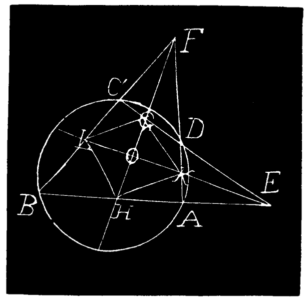
Also solved by *PROF. F. E. MILLER*, and *CHAS. C. CROSS*. Prof. Cooper D. Schmitt did not solve the problem but gave several references where solutions are given. Prof. J. Scheffer gave a short historical note on the problem.

In a future issue of the MONTHLY, we expect to publish a somewhat exhaustive discussion of this very interesting problem.

90. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

The bisectors of the angles of the opposite sides (produced) of an inscribed quadrilateral cut the sides at the angular points of a rhombus.

Solution by G. I. HOPKINS, A. M., Professor of Mathematics in High School, Manchester, N. H.; J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.; J. W. SCROGGS, Principal of Rogers Academy, Rogers, Ark.; NELSON L. RORAY, Professor of Mathematics, South Jersey Institute, Bridgeton, N. J.; HENRY N. DAVIS, Providence, R. I.; ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Ia.; and the PROPOSER.



In the triangles AEK and LEC, $\angle AEG=\angle LEC$, $\angle EAK=\angle LCE$.

$\therefore$ $\angle EKA=\angle ELC$. $\therefore$ $\angle DKL=\angle CLK$.

$\therefore$ FH is perpendicular to KL at its middle point. Similarly, EL is perpendicular to GH at its middle point.

$\therefore$ In the right triangles KOG, KOH, $KO=KO$, $GO=OH$.

$\therefore$ $KG=KH$. Similarly $KG=GL=LH$.

$\therefore$ $KGLH$ is a rhombus.

This problem was also solved in a similar manner by *E. T. BUSH* and *S. L. ROWAN*, of the Freshman Class of the University of Mississippi; *P. S. BERG*, *W. H. DRANE*, *F. R. HONEY*, *E. R. ROBBINS*, *B. F. SINE*, *J. SCHEFFER*, and *J. F. TRAVIS*.

CALCULUS.

70. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Graduate Fellow in Mathematics in Vanderbilt University, P. O., Lynnville, Tenn.

$$\text{Prove } \int_0^\infty \frac{\cos ax}{1+x^{2n}}dx=-i\frac{\pi}{2n}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\omega^{2r-1}}$$

where n is an integer, a is positive, and ω is $e^{i(\pi\ 2n)}$.

Solution by the PROPOSER.

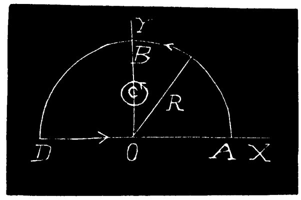
Consider the integral $\int\frac{e^{iay}}{1+y^{2n}}dy$, where a is real and positive. The poles given by $y^{2n}=-1$ or $z=i^{1/n}=\cos(\pi/2n)+i\sin(\pi/2n)=e^{i(\pi/2n)}=\omega$ (say).

It is evident that all the roots of $y^n=i$ are given by ω^{2r-1}, where r may have the values 1, 2, 3, ...n.

Hence $y=\omega^{2r-1}$. About the origin O as center describe a semi-circle ABD with a very large radius, limited by the axis of X (see figure.) About the poles c, which correspond to the points $y^n=i$, describe circles with a very small radius ρ. The proposed function being holomorphic in the portion of the plane lying between the circumferences ρ and the contour $ABDA$, the integrals

$$\int \frac{e^{iay}}{1+y^{2n}}dy$$

relative to the circles and the boundary $ABDA$ are equal. For points on the circles ρ $y=\omega^{2r-1}+\rho e^{i\theta}$, and the integral becomes



$$\int_0^{2\pi} \frac{e^{ia(\omega^{2r-1}+\rho e^{i\theta})}}{1+(\omega^{2r-1}+\rho e^{i\theta})} i\rho e^{i\theta}d\theta, = i\int_0^{2\pi} \frac{e^{ai\omega^{2r-1}}}{2n\,\omega^{(2r-1)(2n-1)}}d\theta$$

(when ρ becomes infinitesimal),

$$=-i\int_0^{2\pi} \frac{\omega^{2r-1}e^{ai\omega^{2r-1}}}{2n}d\theta, = -i\frac{\pi}{n}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\omega^{2r-1}}.$$

For points on the semi-circle ABD, $y=R\cos\theta+iR\sin\theta$, and the integral becomes,

$$i\int_0^{\pi} \frac{Re^{i\theta}e^{aR(\cos\theta+i\sin\theta)}}{1+R^{2n}e^{i2n\theta}}d\theta,$$

which is evidently equal to zero when $R=\infty$, and we have left the integral along DA, which is

$$\int_{-\infty}^{\infty} \frac{e^{aix}}{1+x^{2n}}dx = -i\frac{\pi}{n}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\,\omega^{2r-1}} = \int_{-\infty}^{0}\frac{e^{aix}dx}{1+x^{2n}}+\int_0^{\infty}\frac{e^{aix}dx}{1+x^{2n}}$$

$$=\int_0^{\infty}\frac{e^{aix}+e^{-aix}}{1+x^{2n}}dx, = 2\int_0^{\infty}\frac{\cos ax}{1+x^{2n}}dx.$$

Therefore $\displaystyle\int_0^{\infty}\frac{\cos ax\,dx}{1+x^{2n}} = -i\frac{\pi}{2n}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\omega^{2r-1}}$.

Is this result correst ? Forsyth gives, on page 41 of his *Theory of Functions*, the integral

$$\int_{-\infty}^{\infty}\frac{\cos ax\,dx}{1+x^{2n}} = -i\frac{\pi}{2n}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\omega^{2r-1}}.$$

[Prof. Zerr remarks that the result is correct, as is easily seen from the following:

Let $\cos ax/(1+x^{2n})=f(x)$.

$$\int_{-\infty}^{\infty} f(x)dx=\int_0^{\infty} f(x)dx+\int_{-\infty}^{0} f(x)dx=\int_0^{\infty} f(x)dx-\int_{\infty}^{0} f(x)dx=\int_0^{\infty} f(x)dx$$

$$+\int_0^{\infty} f(x)dx=2\int_0^{\infty} f(x)dx.$$

$$\therefore \int_0^{\infty} f(x)dx=\tfrac{1}{2}\int_{-\infty}^{\infty} f(x)dx,=-i\frac{\pi}{2n}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\omega^{2r-1}}.]$$

71. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Form the differential equation of the third order, of which $y=c_1e^{2x}+c_2e^{-3x}+c_3e^{x}$ is the complete primitive.

I. Solution by EDGAR ODELL LOVETT, Ph. D., Princeton University, Princeton, N. J.

1°. This problem is a familiar one to students of differential equations. The original primitive together with the results of three successive differentiations, may be written

$$\begin{aligned}
&y-e^{2x}c_1-e^{-3x}c_2-e^{x}c_3=0,\\
&y'-2e^{2x}c_1+3e^{-3x}c_2-e^{x}c_3=0,\\
&y''-4e^{2x}c_1-9e^{-3x}c_2-e^{x}c_3=0,\\
&y'''-8e^{2x}c_1+27e^{-3x}c_2-e^{x}c_3=0\,;
\end{aligned}$$

where $y'\equiv\dfrac{dy}{dx}$, $y''\equiv\dfrac{d^2y}{dx^2}$, $y'''\equiv\dfrac{d^3y}{dx^3}$.

The above is a system of linear and homogeneous equations in the quantities 1, $e^{2x}c_1$, $e^{-3x}c_2$, and $e^{x}c_3$, hence the determinant of their coefficients vanishes, that is

$$\begin{vmatrix} y & 1 & 1 & 1\\ y' & 2 & -3 & 1\\ y'' & 4 & 9 & 1\\ y''' & 8 & -27 & 1\end{vmatrix}\equiv\begin{vmatrix} y & 1 & 1 & 1\\ y'-y & 1 & -4 & 0\\ y''-y & 3 & 8 & 0\\ y'''-y & 7 & -28 & 0\end{vmatrix}\equiv4\begin{vmatrix} y'-y & 1 & 1\\ y''-y & 3 & -2\\ y'''-y & 7 & 7\end{vmatrix}\equiv0\,;$$

whence

$$\begin{vmatrix} y'-y & 1 & 0\\ y''-y & 3 & -2\\ y'''-y & 7 & 0\end{vmatrix}\equiv2\begin{vmatrix} y'-y & 1\\ y'''-y & 7\end{vmatrix}=0\,;$$

or finally

$$y'''-7y'+6y=0$$

is the differential equation of the third order whose complete primitive is

$$y - ae^{2x} - be^{-3x} - ce^{x} = 0.$$

2°. If the problem be generalized and the complete primitive taken in the form

$$y - ae^{mx} - be^{-(m+n)x} - ce^{nx} = 0,$$

the corresponding differential equation of the third order is readily found to be

$$y''' - (m^2 + mn + n^2)y' + mn(m+n)y = 0.$$

The values $m=2$ and $n=1$ give the original problem.

3°. If the problem be completely generalized and the original primitive taken in the form

$$y - ae^{px} - be^{qx} - ce^{rx} = 0,$$

the differential equation is

$$y''' - (p+q+r)y'' + (pq + qr + rp)y' - pqry = 0.$$

Putting $p+q+r=0$ we have the second case above. If in addition to $p+q+r=0$, $p=2$ and $q=1$, the first particular case appears again.

II. Solution by **WALTER HUGH DRANE, A. M.**, Professor of Mathematics, Jefferson Military College, Washington, Miss.

(1) $y = c_1e^{2x} + c_2e^{-3x} + c_3e^{x}$. Differentiate (1).

(2) $\dfrac{dy}{dx} = 2c_1e^{2x} - 3c_2e^{-3x} + c_3e^{x}$. Subtract (1) from (2).

(3) $\dfrac{dy}{dx} - y = c_1e^{2x} - 4c_2e^{-3x}$. Differentiate (3).

(4) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 2c_1e^{2x} + 12c_2e^{-3x}$. Subtract twice (3) from (4).

(5) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 20c_2e^{-3x}$. Differentiate (5).

(6) $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = -60c_2e^{-3x}$. Add 3 times (5) to (6).

(7) $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} + 6y = 0$. Q. E. D.

See Johnson's Differential Equations, page 104, example 7.

MECHANICS.

61. Proposed by **WILLIAM HOOVER, A. M., Ph. D.**, Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

A body is suspended from a fixed point by an elastic string, which is stretched to double its natural length when the body is in equilibrium. Find how much the body must be depressed, so that when let go, it may just reach the point of suspension.

I. Solution by the PROPOSER.

Let a', a be the stretched and unstretched lengths of the string. T=the tension, λ=the modulus of elasticity, W=the weight attached, and x=the distance of the latter from the point of suspension at any time t from the beginning of motion.

By Hooke's law, $a'=a[1+(T/\lambda)]$ ....(1).

By the problem, when $a'=2a$, $T=W$, and (1) gives $\lambda=W$.

The forces acting are W and T acting downward and upward ; then

$$\frac{W}{g}\,\frac{d^2x}{dt^2}=W-\frac{W(x-a)}{a} \quad \text{....................(2).}$$

Multiplying both sides of (2) by $2(dx/dt)$ and integrating,

$$\frac{dx^2}{dt^2}=\frac{g}{a}(4ax-x^2)+C \quad \text{......................(3).}$$

When $x=x'$, $(dx/dt)=0$, and (3) gives $C=-(g/a)(4ax'-x'^2)$, and (3) is

$$\frac{dx^2}{dt^2}=\frac{g}{a}[(4ax-x^2)-(4ax'-x'^2)] \quad \text{.................(4).}$$

When the body next comes to rest, $(dx/dt)=0$, and $x=0$, giving $4ax'-x'^2=0$, or $x'=4a$, or $x'=0$. $C=0$ in (3) gives

$$\sqrt{\left(\frac{g}{a}\right)}dt=\frac{dx}{\sqrt{4ax-x^2}} \quad \text{...... .... (5).}$$

Integrating, $t=\left[\sqrt{\left(\frac{a}{g}\right)}\operatorname{versin}^{-1}\left(\frac{x}{2a}\right)\right]_0^{4a}=\pi\sqrt{\left(\frac{a}{g}\right)}$ ....(6),

the *time* for the motion.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

In what follows we neglect the weight of the string, assume Hooke's Law, that the tention of the string is proportional to its extension beyond the natural length, holds throughout the motion.

Let W=weight of body, l=natural length of string, a=extension due to weight W, b=extension at the time body begins to rise after depression, x=extension at any time t, T=corresponding tension of string.

Then by Hooke's Law, $T=Wx/a$ ....(1).

The differential equation of motion is

$m\dfrac{d^2x}{dt^2}=W-T$. But $W=Mg$, $T=Mgx/a$. $\therefore\ \dfrac{d^2x}{dt^2}=(g/a)(a-x)$ ......(2).

$\left(\dfrac{dx}{dt}\right)^2=(g/a)(2ax-x^2)+B$. When $t=0$, $x=b$, $\dfrac{dx}{dt}=0$, $B=(g/a)(b^2-2ab)$.

$$\therefore \left(\frac{dx}{dt}\right)^2=(g/a)(2ax-x^2+b^2-2ab)=v^2.$$

When $x=0$, $v=\sqrt{\frac{gb}{a}(b-2a)}$ = velocity of projection, h = height of projection $=\frac{v^2}{2g}=\frac{b}{2a}(b-2a)$, but $h=2l$ and $a=l$.

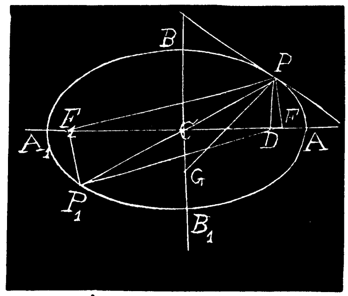
$\therefore\ 2l=(b/2l)(b-2l)$ or $b=l(1+\sqrt{5})$.

62. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A particle of mass m moves in the circumference of an ellipse with constant rate v. It is constrained to move in that circumference by attractive forces in the two foci. To determine the magnitude of these forces,

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let P be the particle mass m, $CA=a$, $CB=b$, $PF=r$, $PF_1=r_1$, $CD=x$, $PD=y$, force along $r=f$, force along $r_1=f_1$, R = normal reaction, X = component of f and f_1 parallel to AC, Y = component of f and f_1 parallel to CB, $x^2/a^2+y^2/b^2=1$, the equation to the ellipse. The equations of motion are



$$m(d^2x/dt^2)=X-R(dy/ds);\ m(d^2y/dt^2)=Y+R(dx/ds)\ldots\ldots(1,2).$$

Now $ds/dt=v$, $\frac{dx}{ds}=\frac{a^2y}{\sqrt{a^4y^2+b^4x^2}}=\frac{ay}{b\sqrt{rr_1}}$, $\frac{dy}{ds}=\frac{b^2x}{\sqrt{a^4y^2+b^4x^2}}=\frac{bx}{a\sqrt{rr_1}}$

$$\left(\frac{dx}{dt}\right)^2=\left(\frac{ds}{dt}\right)^2\frac{a^4y^2}{a^4y^2+b^4x^2}=\frac{a^2v^2y^2}{b^2(a^2-e^2x^2)}=\frac{v^2(a^2-x^2)}{a^2-e^2x^2}.$$

$$\left(\frac{dy}{dt}\right)^2=\left(\frac{ds}{dt}\right)^2\cdot\frac{b^4x^2}{a^4y^2+b^4x^2}=\frac{b^2v^2x^2}{a^2(a^2-e^2x^2)}.$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{a^2v^2x(1-e^2)}{(a^2-e^2x^2)^2}=-\frac{b^2v^2x}{r^2r_1^2},\ \frac{d^2y}{dt^2}=\frac{a^2v^2y}{(a^2-e^2x^2)^2}=\frac{a^2v^2y}{r^2r_1^2}.$$

$$R=(f+f_1)\cos FPG=b^2(f+f_1)/\sqrt{(b^4+a^2e^2y^2)}=(f+f_1)/\sqrt{(rr_1)}.$$

$$X=f_1\cos EPF_1+f\cos EPF=f_1(ae+x)/r_1-f(ae-x)/r.$$

$$Y=f_1\cos DPF_1+f\cos DPF=y(f_1r+fr_1)/rr_1.$$

Substituting in (1) and (2),

$$-ab^2mv^2x=rr_1(arx+a^2er-br)f_1+rr_1(ar_1x-a^2er_1-bx)f\ \ldots\ldots\ldots\ldots\ldots\ (3).$$

$a^2bmv^2=rr_1(br+a)f_1+rr_1(br_1+a)f$ .... (4).

From (3) and (4) since $x=(r_1-r)/2e$,

$$\frac{f_1r}{a^2r-b^2r_1}=\frac{fr_1}{a^2r_1-b^2r}=\frac{abmv^2}{2[(a^2+b^2+a^2be^2)rr_1-2a^2b^2]}.$$

II. **Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

Let ρ=the radius of curvature of the curve at any position, P, of the particle, ϕ=the angle included by either SP or HP, S and H being the foci, and $2a$ =the major axis. Let CD be the semi-diameter conjugate to CP, C being the center of ellipse.

The normal components of the central attractions must together equal the centrifugal force. We may assume the forces in S and H as proportional to some power of CD; and if the absolute intensities of the two forces are equal, say μ,

$$[\mu(CD)^n+\mu(CD)^n]\cos\phi=v^2/\rho \text{ ....} (1).$$

But $$\rho\cos\phi=\frac{SP.HP}{a}=\frac{CD^2}{a} \text{ ....} (2),$$

and (1) is $$2\mu(CD)^n.\frac{CD^2}{a}=v^2 \text{ ....} (3).$$

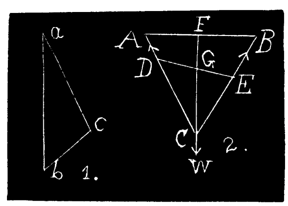
For v=a constant, requires that $n=-2$, showing plainly that the forces vary inversely as the product of the focal distances of the particle.

63. Proposed by A. H. BELL, HILLSBORO, Ill.

From a horizontal support at a distance of ten feet apart, a beam 5 feet long and 10 pounds weight is suspended by ropes attached to each end. The ropes are 3 and 5 feet respectively, in length. Required the angles made by the ropes and horizontal support. Also the stress upon each rope.

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Regarding the bar DE as uniform so that the middle point is the center of gravity, we must have, for equilibrium, the three forces AD, BE, GC passing through the same point C, with GC perpendicular to AB.



The $\angle ABC=\angle CDE$, and the $\angle BAC=\angle CED$.

Let $\angle ABC=\theta$, $\angle BAC=\varphi$, stress on $BE=R$, on $AD=R_1$, weight of $DE=W=10$ pounds. Also $AB=10$, $BE=ED=5$, $AD=3$, $AE^2=100+25-100\cos\theta=25+9+30\cos CDE$.

$\therefore$ $130\cos\theta=91$. $\therefore$ $\cos\theta=\frac{91}{130}=.70000$. $\therefore$ $\theta=45°\ 34'\ 23''$.

$DB^2=100+9-60\cos\varphi=25+25+50\cos CED$.

$\therefore$ $110\cos\varphi=59$. $\therefore$ $\cos\varphi=\frac{59}{110}=.53636$. $\therefore$ $\varphi=57°\ 33'\ 50''$.

Let figure 2 represent the force diagram, $ab=W$, $bc=R$, $ac=R_1$, $\angle abc=90^\circ-\theta$, $\angle bac=90^\circ-\varphi$, $\angle acb=\theta+\varphi$.

$\therefore$ $R=W\cos\varphi/\sin(\theta+\varphi)$, $R_1=W\cos\theta/\sin(\theta+\varphi)$.

$\therefore$ $R=5.5078$ pounds, $R_1=7.1881$ pounds.

DIOPHANTINE ANALYSIS.

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61. Proposed by SYLVESTER ROBBINS, North Branch Depot, New Jersey.

Investigate that infinite series of prime, integral, rational scalene triangles where the sides of every term are consecutive numbers; then take the necessary factors from the proper KEY, and by an expeditious method, find in their order the areas of ten initial terms.

Solution by the PROPOSER.

I. The KEY to this series of rational triangles is $\sqrt{3}=\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \frac{265}{153}, \frac{362}{209}, \frac{989}{571}, \frac{1351}{780}, \frac{3691}{2131}, \frac{5042}{2911}, \frac{13775}{7953}, \frac{18817}{10864}, \frac{51409}{29681}, \frac{70226}{40545}, \frac{191861}{110771}, \frac{262087}{151316}$, etc. Regard the mean side as the base, and drop perpendicular from the opposite angle. Let $x=\frac{1}{2}$base. Notice that $x-2$ and $x+2$ are the segments of the base, and $\sqrt{[3(x^2-1^2)]}$ is the altitude of the triangle. Find such values for x as will render $\sqrt{[3(x^2-1^2)]}$ rational.

When $x=1$, 2, 7, 26, 97, 362, 1351, 5042, 18817, 70226, 262087, $\sqrt{[3(x^2-1^2)]}=0$, 3, 12, 45, 168, 627, 2320, 8733, 32592, 121635, 453948, etc.

These values of x are the half-bases of the several triangles. They are also the numerators of the even convergents in the expansion of $\sqrt{3}$. The values of $\sqrt{[3(x^2-1^2)]}$ are the altitudes of the same triangles, respectively, and they are also three times the denominators of the even convergents in the expansion of $\sqrt{3}$. Multiply one-half the base of a triangle by its perpendicular height, or, three times the product of the terms of the nth even convergent, must give the area of the nth triangle in the series.

Thus, $3\times2\times1=6$; $3\times7\times4=84$; $3\times26\times15=1170$; $3\times97\times56=16296$; $3\times362\times209=226974$; $3\times1351\times780=3161340$; $3\times5042\times2911=44031786$; $3\times18817\times10864=613283664$; $3\times70226\times40545=8541939510$; $3\times262087\times151316=118973869476$, etc.

II. Numerators of even convergents in expansion of $\sqrt{3}$: 1, 2, 7, 26, 97, 362, 1351, 5042, 18817, 70226, etc. Then $\frac{1}{8}(7^2-1^2)=6$; $\frac{1}{8}(26^2-2^2)=84$; $\frac{1}{8}(97^2-7^2)=1170$; $\frac{1}{8}(362^2-26^2)=16296$; $\frac{1}{8}(1351^2-97^2)=226974$; etc.

III. Denominators of even convergents: 1, 4, 15, 56, 209, 780, 2911, etc. $\frac{3}{8}(4^2-0)=6$; $\frac{3}{8}(15^2-1^2)=84$; $\frac{3}{8}(56^2-4^2)=1170$; $\frac{3}{8}(209^1-15^2)=16296$; $\frac{3}{8}(780^2-56^2)=226974$; etc.

IV. Let $x=$the half-sum of the three sides of the triangle. Then $\frac{1}{3}x-1$, $\frac{1}{3}x$ and $\frac{1}{3}x+1$ are the remainders.

$(x)[(\frac{1}{3}x)-1](\frac{1}{3}x)[(\frac{1}{3}x)+1]=$square of triangle. $3(x^2-3^2)=$square.

$\sqrt{\{(\frac{1}{3}x^2)[(x^2-3^2)/9]\}}/x=\frac{1}{3}\sqrt{[(x^2-3^2)/3]}$, the radius of inscribed circle.

Put $x=y+6$; then $3[(y+6)^2-3^2]=$square$=(my+9)^2$.

$3y^2+36y+81=m^2y^2+18my+81$; $3y+36=m^2y+18m$. $y=(18m-36)/(3-m^2)$; and $x=y+6=(18m-18-6m^2)/(3-m^2)$.

Say $m=\sqrt{3}=\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}$, etc. Substitute these values of m in the formula, and we obtain, when

$m=1, x=3$, and $\frac{1}{3}\sqrt{[x^2-3^2)/3]}=0$;
$m=2, x=6$, " " $=1$;
$m=\frac{5}{3}, x=21$, " " $=4$;
$m=\frac{7}{4}, x=78$, " " $=15$;
$m=\frac{19}{11}, x=291$, " " $=56$;
$m=\frac{26}{15}, x=1086$. " " $=209$; etc.

Here the several values of x, the half-sums of three sides, are also three times the numerators of the even convergents in the expansion of $\sqrt{3}$, and the several values of $\frac{1}{3}\sqrt{[(x^2-3^2)/3]}$ are the radii of the inscribed circles, respectively, and also the denominators of the even convergents. Multiplying the half-sum by the radius of the inscribed circle, or taking three times the product of the terms of the nth even convergent in the expansion of $\sqrt{3}$ will give the area of the nth triangle in the series where the common difference in sides is unity.

V. Denominators of odd convergents : 1, 3, 11, 41, 153, 571, 2131, etc. $\frac{3}{4}(3^2-1^2)=6$; $\frac{3}{4}(11^2-3^2)=84$; $\frac{3}{4}(41^2-11^2)=1170$; $\frac{3}{4}(153^2-41^2)=16296$; $\frac{3}{4}(571^2-153^2)=226974$; etc.

VI. Numerators of odd convergents : 1, 5, 19, 71, 265, 989, 3691, etc. $(2\frac{1}{2})^2-(\frac{1}{2})^2=6$; $(9\frac{1}{2})^2-(2\frac{1}{2})^2=84$; $(35\frac{1}{2})^2-(9\frac{1}{2})^2=1170$; $(132\frac{1}{2})^2-(35\frac{1}{2})^2=16296$; $(494\frac{1}{2})^2-(132\frac{1}{2})^2=226974$; etc.

VII. Let $x+1$, x and $x-1$ be the remainders. Then $3x$=the half-sum. $(3x)(x+1)(x)(x-1)$=square of area. $3(x^2-1^2)$=square. Put $x=y+2$. $3[(y+2)^2-1^2]$=square$=(my+3)^2$. $3y^2+12y+9=m^2y^2+6my+9$; $y=(6m-12)/(3-m^2)$; $x=(6m-6-2m^2)/(3-m^2)$. Consent that $m=\sqrt{3}=\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}$, etc. Substitute in formula, and $x=1, 2, 7, 26, 97, 362, 1351$, etc., and the areas of the triangles are as follows : $\sqrt{(3\times2\times1\times0)}=3\times1\times0=0$; $\sqrt{(6\times3\times2\times1)}=3\times2\times1=6$; $\sqrt{(21\times8\times7\times6)}=3\times7\times4=84$; $\sqrt{(18\times27\times26\times25)}=3\times26\times15=1170$; $\sqrt{(291\times98\times97\times96)}=3\times97\times56=16296$; $\sqrt{(1086\times363\times362\times361}=3\times362\times209=226974$, etc.

VIII. Odd convergents : $\sqrt{3}=\frac{1}{1}, \frac{5}{3}, \frac{19}{11}, \frac{71}{41}, \frac{265}{153}$, etc. $6\times1\times1=6$; $(6\times3\times5)-6=84$; $(6\times11\times19)-84=1170$; $(6\times41\times71)-1170=16296$; $(6\times153\times265)-16296=226974$; etc.

IX. Relation of areas : $14\triangle_n-\triangle_{n-1}=\triangle_{n+1}$; $M=14$. 6 ; $14\times6=84$; $(14\times84)-6=1170$; $(14\times1170)-84=16296$; $(14\times16296)-1170=226974$, etc.

X. When x represents the half-base, $2x-1$, $2x$ and $2r+1$ are the sides, $x-2$ and $x+2$ are the segments of the base , $\sqrt{[3(x^2-1^2)}$ is the altitude of the triangle ; and $x\sqrt{[3(x^2-1^2)]}/(4x^2-1^2)$ is the half-sine of angle opposite base. Giving x same values as in I, the half-sines are found to be $\frac{2}{5}=(2\times1)/(2^2+1^2)$; $\frac{28}{65}=(7\times4)/(7^2+4^2)$; $\frac{390}{901}=(26\times15)/(26^2+15^2)$; $\frac{5432}{12545}=(97\times56)/(97^2+56^2)$; $75658/(362^2+209^2)=(362\times209)/(362^2+209^2)$; etc.

Multiply the product of two sides by one-half sine of included angle for area of triangle.

$[(2\times2)-1][(2\times2)+1][(2\times1)/(2^2+1^2)]=3\times5\times\frac{2}{5}=3\times1\times2=6$; $[(2\times7)-1]$

$[(2\times7)+1][(4\times7)/(4^2+7^2)]=13\times15\times\frac{28}{65}=3\times4\times7=84$; $[(2\times26)-1][(2\times26+1]$ $[(15\times26)/(15^2+26^2)]=51\times53\times\frac{390}{901}=3\times15\times26=1170$; $[(2\times97)-1][(2\times97+1]$ $[(56\times97)/(56^2+97^2)]=193\times195\times\frac{5432}{12545}=3\times56\times97=16296$; $[(2\times362)-1][(2\times 362)+1]$ $[(209\times362)/(209^2+362^2)] = 723\times725\times\frac{75658}{174725}= 3\times209\times362=226974$; etc.

Here it should be noticed that in canceling both sides the denominator of the half-sine disappears, and three times the product of the terms of the nth even convergent in the expansion of $\sqrt{3}$ brings the area to light ; also observe, since sides and denominator fall out of view, and factor 3 stands constant, the area must be determined by the numerator of these half-sines, and *this* series may be continued by use of Magic $M=14$; $14\times2=28$; $(14\times28)-2=390$; $(14\times390)-28=5432$; $(14\times5432)-390=75658$; etc.

Also solved by *JOSIAH H. DRUMMOND*, *M. A. GRUBER*, and *G. B. M. ZERR*.

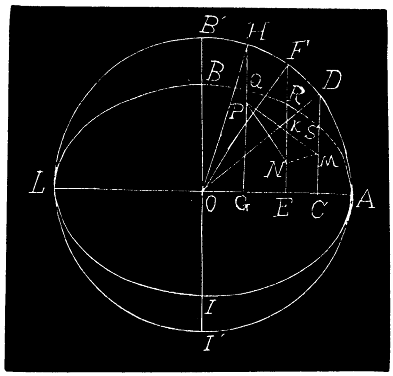
AVERAGE AND PROBABILITY.

60. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Four points are taken at random within an ellipse. What is the chance that they form a reentrant quadrilateral ?

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

We will solve this problem for the quadrant, the semi-ellipse, and the whole ellipse.



Let $ABLI$ be the ellipse, and $AB'LI'$ the circumscribing circle ; M, N, P the three random points ; through M, N, P draw CD, EF, GH perpendicular to AO, EF intersecting MP at K. The triangle will pass through all the possible variations by considering only those relative positions of the points in which CD lies to the right of GH, and EF between CD and GH.

If the fourth point falls anywhere on the triangle formed by joining the points M, N, P, the quadrilateral thus formed will be reentrant.

Let $OA=a$, $OB=b$, $GP=x$, $CM=y$, $EN=z$, $GQ=x'$, $CS=y'$, $ER=z'$, $EK=z''$, $\angle GOH=\theta$, $\angle COD=\varphi$, $\angle EOF=\psi$.

Then we have $x'=b\sin\theta$, $y'=b\sin\varphi$, $z'=b\sin\psi$, $v=1/(\cos\varphi-\cos\theta)$, $z''=v[x(\cos\varphi-\cos\psi)+y(\cos\psi-\cos\theta)]$.

Area $MNP=\frac{1}{2}a[x(\cos\varphi-\cos\psi)+y(\cos\psi-\cos\theta)+z(\cos\theta-\cos\varphi)]=u$, when $z<z''$. Area $MNP=\frac{1}{2}a[x(\cos\psi-\cos\varphi)+y(\cos\theta-\cos\psi)+z(\cos\varphi-\cos\theta)]=u_1$, when

$z>z''$. An element of surface at M is $a\sin\varphi d\varphi dy$, at N it is $a\sin\psi d\psi dz$, at P it is $a\sin\theta d\theta dx$.

The limits of θ are (for quadrant) 0 and $\frac{1}{2}\pi$; of φ, 0 and θ ; of ψ, φ and θ ; of x, 0 and x'; of y, 0 and y'; of z, 0 and z'', and z'' and z'.

Hence the required average area is,

$$\triangle=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}\int_0^{y'}\left(\int_0^{z''}udz+\int_{z''}^{z'}u_1dz\right)a\sin\theta d\theta a\sin\varphi d\varphi a\sin\psi d\psi dxdy}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\phi}^{\theta}\int_0^{x'}\int_0^{y'}\int_0^{z'}a\sin\theta d\theta a\sin\varphi d\varphi a\sin\psi d\psi dxdydz}$$

$$=\frac{384}{\pi^3b^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}\int_0^{y'}\left(\int_0^{z''}udz+\int_{z''}^{z'}u_1dz\right)\sin\theta\sin\varphi\sin\psi d\theta d\varphi d\psi dxdy$$

$$=\frac{96a}{\pi^3b^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}\int_0^{y}\{[x(\cos\varphi-\cos\psi)+y(\cos\psi-\cos\theta)]^2+[x(\cos\varphi-\cos\psi)$$

$$+y(\cos\psi-\cos\theta)+b\sin\psi(\cos\theta-\cos\psi)]^2\}\sin\theta\sin\varphi\sin\psi v d\theta d\varphi d\psi dxdy$$

$$=\frac{32a}{\pi^3b^2}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}[6x^2\sin\varphi(\cos\varphi-\cos\psi)^2+6bx\sin^2\varphi(\cos\varphi-\cos\psi)(\cos\psi$$

$$-\cos\theta)+6bx\sin\varphi\sin\psi(\cos\varphi-\cos\psi)(\cos\theta-\cos\varphi)+2b^2\sin^3\varphi(\cos\psi-\cos\theta)^2$$

$$+3b^2\sin\varphi\sin^2\psi(\cos\theta-\cos\varphi)^2+3b^2\sin^2\varphi\sin\psi(\cos\theta-\cos\varphi)(\cos\psi-\cos\theta)]$$

$$\times\sin\theta\sin\varphi\sin\psi v d\theta d\varphi d\psi dx.$$

$$\triangle=\frac{32ab}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}[2\sin^3\theta\sin\varphi(\cos\varphi-\cos\psi)^2+2\sin\theta\sin^3\varphi(\cos\psi-\cos\theta)^2$$

$$+3\sin^2\theta\sin^2\varphi(\cos\varphi-\cos\psi)(\cos\psi-\cos\theta)+3\sin\theta\sin\varphi\sin^2\psi(\cos\theta-\cos\varphi)^2$$

$$+3\sin^2\theta\sin\varphi\sin\psi(\cos\varphi-\cos\psi)(\cos\theta-\cos\varphi)$$

$$+3\sin\theta\sin^2\varphi\sin\psi(\cos\psi-\cos\theta)(\cos\theta-\cos\varphi)]\sin\theta\sin\varphi\sin\psi v d\theta d\varphi d\psi$$

$$=\frac{16ab}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}[4\sin^2\theta\cos^2\varphi+4\sin^2\varphi\cos^2\varphi+4\sin^2\theta\cos^2\theta+4\sin^2\varphi\cos^2\theta$$

$$+\sin^2\theta\cos\theta\cos\varphi+\sin^2\varphi\cos\varphi\cos\theta-6\sin\theta\cos\theta\sin\varphi\cos\varphi$$

$+6\cos^3\theta\cos\varphi+6\cos\theta\cos^3\varphi+12+6\cos^2\theta+6\cos^2\varphi-36\cos\theta\cos\varphi$

$-12\sin\theta\sin\varphi-9(\theta-\phi)\sin\theta\cos\varphi+9(\theta-\phi)\sin\varphi\cos\theta]\sin^2\theta\sin^2\varphi d\theta d\varphi$

$$=\frac{8ab}{9\pi^3}\int_0^{\frac{1}{2}\pi}(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta-45\sin\theta\cos\theta$$

$$-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta)\sin^2\theta d\theta$$

$$=\frac{ab}{\pi}\left(\frac{35}{12}+\frac{16}{3\pi}-\frac{131}{3\pi^2}\right).$$

For the semi-ellipse above the major axis, the limits of θ are 0 and π, and those of the other variables the same as above. The number of ways the three points can be taken in the semi-ellipse is eight times the number of ways in a quadrant, and hence we get

$$\triangle_1=\frac{ab}{9\pi^3}\int_0^{\pi}(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta-45\sin\theta\cos\theta$$

$$-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta)\sin^2\theta d\theta=\frac{ab}{\pi}\left(\frac{35}{24}-\frac{32}{3\pi^2}\right).$$

For the limits of θ are 0 and 2π, and the points can be taken eight times the number of ways in semi-ellipse. Hence

$$\triangle_2=\frac{ab}{72\pi^3}\int_0^{2\pi}(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta-45\sin\theta\cos\theta$$

$$-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta)\sin^2\theta d\theta=35ab/48\pi.$$

Let C, C_1, C_2 be the respective chances required.

$$C=\frac{4\triangle}{\pi ab}=\frac{4}{\pi^2}\left(\frac{35}{12}+\frac{16}{3\pi}-\frac{131}{3\pi^2}\right);\quad C_1=\frac{4\triangle_1}{\pi ab}=\left(\frac{35}{42}-\frac{32}{3\pi^2}\right);$$

$$C_2=\frac{4\triangle_2}{\pi ab}=\frac{35}{12\pi^2}.$$

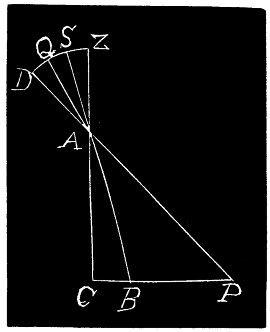
MISCELLANEOUS.

58. Proposed by EDMUND FISH, Hillsboro, Ill.

The longest noonday winter shadow of an upright object is found to be seven times as long as the shortest summer shadow of the same object. Required the latitude of the place.

I. Solution by S. HART WRIGHT, A. M., M. D., Ph. D., Penn Yan, N. Y.

In the right plane triangles ABC and APC, let the vertical AC(=unity) be the rod that casts a shadow from C to B, and from C to P, when the sun is at S and D. Extend CA to the zenith Z, BA to S and PA to D. Bisect DAS with QA. Let $\angle BAC=\chi=ZAS$. DAS is double the obliquity of the ecliptic$=2\delta=\angle PAB=46^\circ\ 54'\ 30''=v$. $DAQ=SAQ=\delta$, and Q must be on the equator, and QAZ=the required latitude$=\lambda$. $CP=7CB$. Put $7=m$, and $\tan^{-1}m=\beta=81^\circ\ 52'\ 12''$. We have $CB=\tan\chi$, and $CP=\tan(\chi+v)$, and $m\tan\chi=\tan(\chi+v)$, a trigonometric equation, from which we derive $\sin(2\chi\pm v)=\cot(\beta\mp45^\circ)\sin v$. Four values of χ result, the upper signs giving the only acceptable value of $\chi=14^\circ\ 57'\ 30''$. The other signs make the $\angle PAC>90^\circ$. Now $\lambda=\chi+\delta=38^\circ\ 24'\ 45''$ north or south, as the seasons are interchangeable on each side of the equator.



II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let AC be the upright object length$=l$, CP the earth, SB the summer sun, DP the winter sun. Let ϕ=latitude, δ=north, $-\delta$=south declination of the sun, and let the winter shadow be n times as long as the summer shadow.

Then $\angle CAB=(\phi-\delta)$, $\angle CAP=(\phi+\delta)$, $CP=n.CB$, $CP=l\tan(\phi+\delta)$, $CB=l\tan(\phi-\delta)$.

$\therefore\ l\tan(\phi+\delta)=nl\tan(\phi-\delta)$.

$\therefore\ (n+1)\tan^2\phi\tan\delta-(n-1)\tan\phi\sec^2\delta+(n+1)\tan\delta=0$.

$$\tan^2\phi-\frac{2(n-1)\tan\phi}{(n+1)\sin2\delta}+1=0.$$

$$\therefore\ \tan\phi=\frac{(n-2)\pm\sqrt{[(n-1)^2-(n+1)^2\sin^2 2\delta]}}{(n+1)\sin2\delta}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

Now $\delta=23^\circ\ 27'\ 30''$, $n=7$. $\therefore\ \tan\phi=.793428$ or 1.26035.

$\therefore\ \phi=38^\circ\ 25'\ 46''$ or $51^\circ\ 34'\ 14''$.

The two values of $\tan\phi$ are equal when $n=(1+\sin2\delta)/(1-\sin2\delta)$.

$\therefore\ n=6.4118$, $\phi=45^\circ$. When $\phi=\delta$, and $\phi+\delta=90^\circ$, n is infinite.

In the first case the summer shadow is zero and the winter shadow is fi-

nite ; in the second case, the winter shadow is infinite and the summer shadow is finite.

In formula (A), δ and n can have any values within proper limits.

Also solved by *W. W. LANDIS*, and *J. SCHEFFER*.

59. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

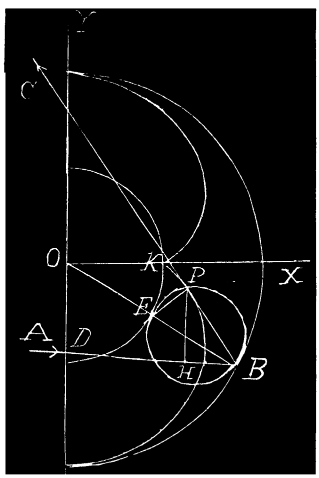
When a cylindrical china jar, standing upon the ground, receives the sun's rays obliquely, a bright curve is observed to form itself at the bottom of the jar, and it is found that the shape and dimensions of this curve are not affected by the varying elevations of the sun: account for this latter circumstance, and determine the nature of the bright curve. [From *Parkinson's Optics.*]

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Mississsppi.

All rays striking any element of the cylindrical surface lie in a vertical plane. Their reflections form the other face of the dihedral angle whose bisector passes through the axis of the cylinder. These reflected rays intersect the base of the cylinder in a straight line. There is thus formed a system of lines, and the bright curve observed is their envelope. The altitude of the sun does not affect the position of the vertical planes; and, therefore, the intersections with the bottom of the jar are unchanged, and the continual intersection of the consecutive lines so formed produces a curve invariable as to its shape and size.

The bright curve is the caustic by reflection for the circle, the incident rays being parallel. The following general property of caustics by reflection for parallel rays is established in *Price's Infinitesimal Calculus*: "The distance from the incident point in the reflecting curve to the point of intersection of two consecutive reflected rays, is equal to one-fourth of the chord of the circle of curvature at the point of incidence which is parallel to the incident ray."

A. Take the center of the circle as the origin, the X-axis parallel to the incident rays, the Y-axis perpendicular to them. Let AB be an incident ray, BC its reflection, the angle between them being 2θ. Take BP along BC equal to one-half of DB, D being the intersection of AB with the Y-axis. Then, according to the principle quoted above, P is a point of the caustic. To find the locus of P: Draw OB, denoting it by a. From P drop a perpendicular to AB meeting it at H. Denoting the coördinates of P by x and y,



$$x=DB-HB=a\cos\theta-\tfrac{1}{2}a\cos\theta\cos2\theta,$$

$$y=OD-PH=a\sin\theta-\tfrac{1}{2}a\cos\theta\sin2\theta.$$

From these $x=\frac{3}{4}a\cos\theta-\frac{1}{4}a\cos3\theta$,

$$\text{and } y=\tfrac{3}{4}a\sin\theta-\tfrac{1}{4}a\sin3\theta.$$

These may be written

$x=(\frac{1}{2}a+\frac{1}{4}a)\cos\theta-(\frac{1}{4}a)\cos[(\frac{1}{2}a+\frac{1}{4}a)/(\frac{1}{4}a)]\theta$,

and $y=(\frac{1}{2}a+\frac{1}{4}a)\sin\theta-(\frac{1}{4}a)\sin[(\frac{1}{2}a+\frac{1}{4}a)/(\frac{1}{4}a)]\theta$.

These are the well-known equations of an epicycloid, the radii of the fixed and rolling circles being $\frac{1}{2}a$ and $\frac{1}{4}a$ respectively.

B. The following geometrical solution is very much like one given in *Wood's Optics*, and was suggested by it.

Referring to the same figure, erect at P a perpendicular to CB, meeting OB at E. Comparing the similar triangles EPB and ODB, $ED : DB=BP : BD =1 : 2$.

If, then, upon EB, the half of OB, as diameter, a circumference be drawn its intersection with CB will be a point of the caustic. With O as center and EO as radius describe a semi-circle, intersecting the X-axis at K.

The $\angle EOK=\theta$, and arc $EK=\frac{1}{2}a\theta$.

Also, since $\angle EBP=\theta$, the angle at the center measured by arc $EP=2\theta$; and arc $EP=\frac{1}{4}a.2\theta=\frac{1}{2}a.\theta$.

Hence arc EP=arc EK.

The locus of P is, therefore, generated by the circle EPB rolling on the circle EK, the points P and K being originally in contact.

Of course the problem may be solved without assuming the property quoted from *Price*. In *Rice and Johnson's Differential Calculus* an excellent solution is outlined.

Also solved by *C. W. M. BLACK*, *S. H. WRIGHT*, and *B. F. FINKEL*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

97. **Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics. Curry University, Pittsburg, Pa.**

In what time will $4000 amount to $5134.96, interest at 6% payable annually ?

*** Solutions of these problems should be sent to B. F. Finkel, not later than July 10.

GEOMETRY.

97. **Proposed by CHAS. C. CROSS, Libertytown, Md.**

Prove by pure geometry: The radius of a circle drawn through the centers of the inscribed and any two escribed circles of a triangle is double the radius of the circumscribed circle of the triangle.

98. **Proposed by EDW. R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School, Lawrenceville, N. J.**

Construct a circle which shall pass through two given points and touch a given circle, (1) when the distance between the points is less than the diameter of the circle, and (2) when it is greater.

*** Solutions of these problems should be sent to B. F. Finkel, not later than July 10.

MECHANICS.

69. Proposed by B. F. FINKEL, A. M.,M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A rough sphere of radius a and radius of gyration K, capable of rotating about its center, is initially at rest; another sphere of $1/n$ the mass and of radius b, and radius of gyration k, is placed gently on it, having initially an angular velocity ω about the common normal which makes an acute angle α with the vertical drawn upwards. Prove that the second sphere will not roll off provided

$$\omega^2 > \frac{2\mu(a+b)g}{(3\mu+1)b^2}[(3\mu+1)^2-4\mu^2\cos^2\alpha]\sec\alpha,\text{ where } \mu=a^2/nK^2+b^2/k^2.$$

[From *Routh's Rigid Dynamics.*]

70. Proposed by CHAS. E. MEYERS, Canton, Ohio.

A homogeneous sphere, radius r, having an angular velocity ω, gradually contracts by cooling. What will be the angular velocity at the instant the radius becomes $\frac{1}{2}r$?

⁂ Solutions of these problems should be sent to B. F. Finkel, not later than August 10.

DIOPHANTINE ANALYSIS.

68. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find a *general* value for p in the expression $4p+1$=the sum of two squares.

69. Proposed by JOSIAH H. DRUMMOND, LL. D., Counselor at Law, Portland, Me.

Two right angled triangles have the same base which is a mean proportional between the two perpendiculars: find a general solution, that will give integral values for all the sides of both triangles.

70. Proposed by PROF. CHARLES CARROLL CROSS, Libertytown, Md.

Give methods for decomposing numbers into squares, cubes, or biquadrates and show that 61×200^3 is the sum of ten cube numbers and that 844933 is the sum of eleven biquadrates in thirteen different ways. [From *The Mathematical Magazine*, Vol. II, No. 10.]

⁂ Solutions of these problems should be sent to J. M. Colaw, not later than August 10.

AVERAGE AND PROBABILITY.

65. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

What is the average rate of the sun's motion in declination from the equator to the solstices?

66. Proposed by REV. W. ALLEN WHITWORTH, A. M.

A rod 9 feet long is to be divided into three parts, of which A is to have the largest, B the next, and C the smallest. If the two fractures are made at random, A's, B's, and C's expectations will be respectively 66, 30, and 12 inches. But, if one fracture be made at random and the larger portion of the rod be then divided at random, their expectations will be 64, 31, and 13 inches.

67. **Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

A person writes n letters and addresses n envelopes; if the letters are placed in the envelopes at random, what is the probability that every letter goes wrong? [From *Hall and Knight's Higher Algebra.*]

*** Solutions of these problems should be sent to B. F. Finkel, not later than August 10.

MISCELLANEOUS.

62. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

How many bushels of wheat will a conical bin 8 feet in diameter at base and 12 feet high, hold, if part of the bin is cut off by a plane parallel to the side and passing through the center of the base?

63. **Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.**

Show that the path of a projectile moving with a constant velocity is an inverted catenary of equal strength.

*** Solutions of these problems should be sent to J. M. Colaw, not later than August 10.

EDITORIALS.

The degree of Civil Engineer has been conferred on our valued contributor, Fremont Crane, by the University of Minnesota.

In our next issue will appear an article on Symmetric Functions, by Professor E. D. Roe, Associate Professor of Mathematics in Oberlin College, now at Erlangen, Germany.

Dr. David Eugene Smith, Professor of Mathematics in the Michigan State Normal School, has accepted the Presidency of the New York State Normal School, at Rockport.

In our last issue, the last two pages of the MONTHLY were hurried into print without our having read the proof. That we might have the opportunity of eliminating some of the errors which thus appeared in them, we have had those two pages reprinted and bound in the present issue. This, we are sure, will be appreciated by those who wish to have the volumes of the MONTHLY bound.

THE UNIVERSITY OF CHICAGO. During the summer quarter (July 1 to September 23, 1898) the following mathematical courses (four or five hours weekly) will be offered:—By Associate Professor Maschke: Theory of Invariants; Functions of a Complex Variable;—By Assistant Professor Young: Mathematical Pedagogy (to August 15); Culture Calculus; Plane Trigonometry (to August

15); College Algebra (after August 15, ten hours weekly):—By Dr. Hancock: Calculus of Variations; Theory of Equations (advanced):—By Dr. Slaught: Advanced Integral Calculus; Plane Analytics:—By Dr. Miller (of Cornell University): Seminar in Permutation Groups (to August 15).

General circulars of information and programmes of the Department of Mathematics may be obtained from the Examiner of the University.

BOOKS AND PERIODICALS.

Lessons in Elementary Mechanics. Introductory to the study of Physical Science, designed for the use of Schools and of Candidates for the London Matriculation and other examinations, with numerous exercises. By Sir Philip Magnus, author of Class-Book on Hydrostatics and Pneumatics, etc. New Edition, Rewritten and Enlarged in 1892. Fourth Thousand. 8vo Cloth, 377 pages. Price, \$2.25. New York: Longmans, Green & Co.

At the present time, there is a very great demand for a thorough knowledge of the principles of mechanics. The present high civilization, while due to many influences, is, however, more largely due to the increased knowledge of the laws of nature and her modes of operation. The little work before us is designed to give the beginner a thorough understanding of all the elementary principles of mechanics. The style is lucid, the work compact and comprehensive, the illustrative problems well chosen, and the numerous exercises carefully graded. B. F. F.

A Discussion of the Cyclic Quadrilateral. By J. C. Gregg, A. M., Superintendent of Schools, Brazil, Ind. Pamphlet, 5 pages.

This is a very good discussion, and is illustrated by two figures. B. F. F.

Annals of Mathematics. Edited by Wm. H. Echols. Published under the auspices of the University of Virginia. Price, \$2.00 per year in advance.

The May number contains an article on the Calculus of Variation, by Dr. Harris Hancock; Notes on Some Points in Theory of Linear Differential Equations, by Dr. Maxime Bocher; On Triple Focus of a Cartesian, by Dr. Carl C. Engberg; Direct Derivation of Ordinary Canonical System of Elliptic Elements Employed in the Problem of Three Bodies. The June number: The Law of the Mean and the Limits of $0/0$, ∞/∞, by Prof. W. F. Osgood; Certain Invariants of a Quadrangle by Projective Transformations, by Dr. Edgar Odell Lovett; The General Transformation of the Group of Euclidean Movements, by Prof. J. M. Page; Concomitant Binary Forms in Terms of the Roots, by Miss Annie L. MacKennon. B. F. F.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, \$2.50 per year, in advance. Single number, 25 cents. The Review of Reviews Co., New York.

"The Progress of the World," the editorial department of the *Review of Reviews*, gives not only a complete history of the war to date, but also a full discussion of the collateral issues involved, such as the acquisition of new territory and the proposed alliances with other powers. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, \$1.00 per year in advance. Irvington-on-the-Hudson, New York.

In the June number is a most excellent article on Liquid Air. This article alone is worth a years subscription to the magazine. B. F. F.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. JUNE-JULY, 1898. No. 6-7.

NOTE ON A FORMULA OF SYMMETRIC FUNCTIONS.

By E. D. ROE, JR., Associate Professor of Mathematics in Oberlin College, Oberlin, Ohio.

On page 8 of the German translation of Faà di Bruno's Formes Binaires, one reads : "The function $\phi_k[=\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_k{}^{p_k}]$ can be written symbollically in the form of a determinant as follows,

$$\phi_k=\begin{vmatrix} s_{p_1} & s_{(p_2)} & s_{(p_3)} & \cdots\cdots \\ s_{(p_2)} & s_{p_2} & s_{(p_2)} & \cdots\cdots \\ s_{(p_3)} & s_{(p_3)} & s_{p_3} & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

if after the development of the determinant, the symbolic products $s_{(p_1)}\,s_{(p_2)}\ldots.$ are converted into $s_{p_1+p_2+\ldots\ldots}$."

If one endeavors to carry out these instructions, he finds that the statement holds for $k=2$, and $k=3$, but not farther. In the Annali di Tortolini (t. 5, 1854, p. 427-8), a clearer statement of this formula is given by Brioschi, but without proof. Brioschi's statement, slightly changing the notation, is as follows :

If $u_{rr}=s_{p_r}$, $u_{rs}u_{sr}=s_{p_r+p_s}$, $u_{rs}u_{st}u_{tr}=s_{p_r+p_s+p_t}$, and in general if every cycle of a substitution corresponds to a power sum whose index is the sum of those p's whose subscripts enter into the cycle, then the $n\,!$ products $u_{1r_1}u_{2r_2}u_{3r_3}\ldots.u_{nr_n}$, corresponding to the $\begin{pmatrix}1 & 2 & 3\ldots.n \\ r_1 & r_2 & r_3\ldots.r_n\end{pmatrix}$, where the r's form a permutation of the numbers $1\ 2\ 3\ldots.n$, give the terms of $\phi_n=\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots.\alpha_n{}^{p_n}$, which we may write in the form of a determinant as follows :

$$\phi_n = \begin{vmatrix} u_{11} & u_{12} \ldots\ldots & u_{1n} \\ u_{21} & u_{22} \ldots\ldots & u_{2n} \\ \ldots & \ldots & \ldots \\ u_{n1} & u_{n2} \ldots\ldots & u_{nn} \end{vmatrix}$$

It seems desirable to give here a proof of this theorem. One may be given in this manner: We form,

$$s_{p_n} \Sigma \alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_{n-1}^{p_{n-1}} = \sum_{r=1}^{r=n} \alpha_r^{p_n} (\alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_{n-1}^{p_{n-1}}).$$

This gives $\phi_n = - \sum_{r=1}^{r=n-1} \phi_{n-1}^{(r)} + s_{p_n} \phi_{n-1}$, where $\phi_{n-1}^{(r)}$ is an abbreviation for $\Sigma \alpha_r^{p_n} (\alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_{n-1}^{p_{n-1}})$. We assume that the determinant formula is true when $n-1$ roots are involved. Then the preceding equation takes the form

$$\phi_n = - \Sigma \begin{vmatrix} v_{11} & v_{12} \ldots & v_{1n-1} \\ v_{21} & v_{22} \ldots & v_{2n-1} \\ \ldots & \ldots & \ldots \\ v_{n-11} & v_{n-12} \ldots & v_{n-1n-1} \end{vmatrix} + u_{nn} \begin{vmatrix} u_{11} & u_{12} \ldots & u_{1n-1} \\ u_{21} & u_{22} \ldots & u_{2n-1} \\ \ldots & \ldots & \ldots \\ u_{n-11} & u_{n-12} \ldots & u_{n-1n-1} \end{vmatrix}$$

The sum Σ contains $(n-1)$ determinants. Of these the r_{11}th one, which expresses $\phi_{n-1}^{(r)}$, must be similar to ϕ_{n-1}, but with this difference, that everywhere instead of p_r, p_r+p_n must occur. If we denote p_r+p_n by p_r^1, we may say, the symbol connected with $s_{p_r^1}$ must be so constituted as to always express $s_{p_r}+p_n$. Any product which corresponds to a cycle which begins and ends with r, that is, any product commencing with an element of the rth line, and ending with an element of the rth column, like

$$v_{rs} v_{st} \ldots\ldots v_{xy} v_{yr} = s_{p_r^1 + p_s + \ldots\ldots p_x + p_y}$$

can be written in the u notation as

$$u_{nr} u_{rs} u_{st} \ldots\ldots u_{xy} u_{yn} = s_{p_r + p_n + p_s + \ldots\ldots p_x + p_y},$$

since the right members of these two expressions are equal. As far as the products which correspond to the remaining cycles (which cannot involve r and n) are concerned, they may always be written simply with u's instead of v's. It follows: We may write the determinant for $\phi_{n-1}^{(r)}$ with u's instead of v's, provided that only v_{rs} goes over into $u_{nr} u_{rs}$, and v_{yr} into u_{yn}, and obtain exactly the same expansion as before. Writing $u_{nr} u_{r1}$, $u_{nr} u_{r2} \ldots\ldots u_{nr} u_{rn-1}$ for the rth line instead of v_{r1}, $v_{r2} \ldots\ldots v_{rn-1}$, and u_{1n}, $u_{2n} \ldots\ldots u_{n-1n}$ for the rth column instead of v_{1r}, $v_{2r} \ldots\ldots v_{n-1r}$, we have for $-\phi_{n-1}^{(r)}$, after taking out the factor u_{nr},

$$-\phi_{n-1}^{(r)}=-u_{nr}\begin{vmatrix} u_{11} & u_{12}\cdots\cdots u_{1r-1} & u_{1n} & u_{1r+1}\cdots\cdots u_{1n-1} \\ u_{21} & u_{22}\cdots\cdots u_{2r-1} & u_{2n} & u_{2r+1}\cdots\cdots u_{2n-1} \\ \cdots & \cdots & \cdots & \cdots \\ u_{r1} & u_{r2}\cdots\cdots u_{rr-1} & u_{rn} & u_{rr+1}\cdots\cdots u_{rn-1} \\ \cdots & \cdots & \cdots & \cdots \\ u_{n-11} & u_{n-12}\cdots\cdots & u_{n-1n} & \cdots\cdots u_{n-1n-1} \end{vmatrix}$$

In this determinant pass the rth column over the remaining $n-r-1$ columns. The entire sign factor becomes $(-1)^{n-r-1+1}=(-1)^{n-r-2}$. We can now combine our results conveniently as follows :

$$\phi_n=\sum_{r=1}^{r=n}(-1)^{n-r-2}u_{nr}\begin{vmatrix} u_{11} & u_{12}\cdots\cdots u_{1r-1} & u_{1r+1}\cdots\cdots u_{1n} \\ u_{21} & u_{22}\cdots\cdots u_{2r-1} & u_{2r+1}\cdots\cdots u_{2n} \\ \cdots & \cdots & \cdots \\ u_{n-11} & u_{n-12}\cdots u_{n-1r-1} & u_{n-1r+1}\cdots u_{n-1n} \end{vmatrix}$$

if for $r=1$ the first column, and for $r=n$, the last column is suppressed. But the sum of the determinants on the right hand is none other than the expansion of

$$\begin{vmatrix} u_{11} & u_{12}\cdots\cdots u_{1n} \\ u_{21} & u_{22}\cdots\cdots u_{2n} \\ \cdots & \cdots \\ u_{n1} & u_{n2}\cdots\cdots u_{nn} \end{vmatrix}$$

in terms of the elements of the last line.

We have therefore proved that if ϕ_{n-1} can be expressed as a determinant of order $n-1$, ϕ_n can be similarly expressed as a determinant of order n. But the assumption is true when $n-1=1$, and therefore for all greater values.

Attention may be called to the fact that this determinant gives very beautifully the general term of ϕ_n in terms of the s's. Let σ be the number of cycles, involving n_1, n_2, ...u_σ numbers respectively, where $n_2+n_2+\ldots\ldots n_\sigma=n$, into whose product any given substitution of the numbers 1, 2, 3 .....n can be decomposed. The general term of ϕ_n will consist of the products which correspond to the σ cycles of the substitution. The modulus of the substitution will be the product of the moduli of the cycles, viz, $(-1)^{n_1-1+n_2-1+\ldots\ldots n_\sigma-1}=(-1)^{n-\sigma}$. The coefficient of the term will be the number of ways in which the substitution can occur. The first cycle can occur in $(n_1-1)!$ ways, for for the first number the (n_1-1) remaining numbers may be substituted, for the second the (n_1-2) remaining, etc.; for the next to the last, the last remaining unused number, and for the last the first number. Similarly, a second cycle can occur in $(n_2-1)!$ ways, and the two together in $(n_1-1)!\,(n_2-1)!$ ways. We thus get for the general term of ϕ_n, if λ_{n_1}, λ_{n_2}, ... λ_{n_σ} denote the sum of the n_1, n_2, ...n_σ

exponents respectively,

$$(-1)^{n-\sigma}(n_1-1)!\,(n_2-1)!\ldots\ldots(n\sigma-1)!\ {}^s\lambda_{n_1}\ {}^s\lambda_{n_2}\ \ldots\ldots {}^s\lambda_{n\sigma},$$

a result that agrees with that obtained in a somewhat different way on page 8 of the German translation of Faa di Bruno's Formes Binaires.

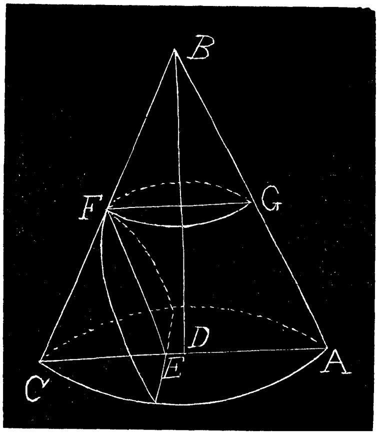
Erlangen, Bavaria, 4 May, 1898.

CONVEX SURFACE AND VOLUME OF CONICAL UNGULÆ.

By G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let $BD=c$, $DE=a$, $\tan DBC=n$, $\cot FEC=m$. Also let $DH=h$, $DC=R$, $HF=r$, $EC=d$, then $c=\dfrac{Rh}{R-r}$, $n=\dfrac{R-r}{h}$, $a=R-d$, $m=\dfrac{r-R+d}{h}$.

Then $x^2+z^2=n^2(c-y)^2$, is the equation of the cone, and $x=my+a$, is the equation of the plane.



$$\left(\frac{dz}{dx}\right)^2=\frac{x^2}{z^2},\quad \left(\frac{dz}{dy}\right)^2=\frac{n^4(c-y)^2}{z^2};$$

$$\therefore \sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}$$

$$=\frac{n(c-y)\sqrt{(1+n^2)}}{\sqrt{[n^2(c-y)^2-x^2]}}.$$

The limits of x are $my+a=x_2$ and $n(c-y)=x_1$; of y, 0 and $\dfrac{nc-a}{m+n}=y'$.

$$\therefore\ S=2\int_0^{y'}\int_{x_2}^{x_1}\frac{n(c-y)\sqrt{(1+n^2)}}{\sqrt{[n^2(c-y)^2-x^2]}}$$

$$=n\sqrt{(1+n^2)}\int_0^{y'}\left[\pi-2\sin^{-1}\left(\frac{a+my}{n(c-y)}\right)\right](c-y)dy$$

$$=nc^2\sqrt{(1-n^2)}\left[\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{a}{nc}\right)\right]$$

$$-n(a+mc)\sqrt{(1+n^2)}\int_0^{y'}\frac{(c-y)dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}.$$

$$n^2(c-y)^2-(a+my)^2=[(mnc+na)^2-(n^2c+am+m^2y-n^2y)^2]/(m^2-n^2).$$

When $m^2>n^2$

$$=(nc+a)(nc-a-2ny).$$

When $m^2=n^2$

$$=[(n^2c+am+m^2y-n^2y)^2-(mnc+na)^2]/(n^2-m^2)$$

When $m^2<n^2$

Let $u=n^2c+am+m^2y-n^2y$.

Then the limits of u are $n^2c+am=u_2$ and $mnc+an=u_1$. When $m^2>n^2$

$$n(a+mc(\sqrt{(1+n^2)}\int_0^{y'}\frac{(c-y)dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=\frac{n(a+mc)\sqrt{(1+n^2)}}{\sqrt{[(m^2-n^2)^3]}}\int_{u_2}^{u_1}\frac{(cm^2+am-u)du}{\sqrt{[(an+mcn)^2-u^2]}}$$

$$=\frac{\pi nm(a+mc)^2\sqrt{(1+n^2)}}{2\sqrt{[(m^2-n^2)^3]}}-\frac{nm(a+mc)^2\sqrt{(1+n^2)}}{\sqrt{[(m^2-n^2)^3]}}\sin^{-1}\left(\frac{n^2c+am}{an+mnc}\right)$$

$$-\frac{n(a+mc)\sqrt{(1+n^2)}}{\sqrt{[(m^2-n^2)^3]}}\sqrt{(an+mnc)^2-(n^2c+am)^2}.$$

$$\therefore\ S=n\sqrt{(1+n^2)}\left[\frac{\pi c^2}{2}-\frac{\pi m(a+mc)^2}{2\sqrt{[(m^2-n^2)^3]}}-c^2\sin^{-1}\left(\frac{a}{nc}\right)\right.$$

$$+\frac{m(a+mc)^2}{\sqrt{[(m^2-n^2)^3]}}\sin^{-1}\left(\frac{n^2c+am}{an+nmc}\right)$$

$$\left.+\frac{(a+mc)}{\sqrt{[(m^2-n^2)^3]}}\sqrt{(an+nmc)^2-(n^2c+am)^2}\right].$$

$$S=\frac{\sqrt{[h^2+(R-r)^2]}}{R-r}\left[\frac{\pi R^2}{2}-\frac{\pi r^2d(r-R+d)}{2\sqrt{[d(d+2r-2R)^3]}}\right.$$

$$+\frac{r^2d(r-R+d)}{\sqrt{[d(d+2r-2R)^3]}}\sin^{-1}\left(\frac{2R-r-d}{r}\right)$$

$$-R^2\sin^{-1}\left(\frac{R-d}{R}\right)+\frac{rd(R-r)}{\sqrt{[d(d+2r-2R)^3]}}\sqrt{r^2-(2R-r-d)^2}.$$

Let $d=2R$. $\therefore\ S=\dfrac{\pi\sqrt{[h^2+(R-r)^2]}}{R-r}\left(R^2-\tfrac{1}{2}\sqrt{Rr}(R+r)\right).$

When $m^2=n^2$, $y'=(nc-a)/2n$.

$$n(a+mc)\sqrt{(1+n^2)}\int_0^{y'}\frac{(c-y)dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=n\sqrt{a+nc}\sqrt{1+n^2}\int_0^{y'}\frac{(c-y)dy}{\sqrt{[nc-a-2ny]}}=\frac{(2nc+a)\sqrt{[(n^2c^2-a^2)(1+n^2)]}}{3n}.$$

$$\therefore\ S=\sqrt{(1+n^2)}\left[\frac{\pi c^2n}{2}-nc^2\sin^{-1}\left(\frac{a}{nc}\right)-\frac{(2nc+a)\sqrt{(n^2c^2-a^2)}}{3n}\right].$$

$$\therefore\ S=\frac{\sqrt{[h^2+(R-r)^2]}}{R-r}\left[\frac{\pi R^2}{2}-R^2\sin^{-1}\left(\frac{R-d}{R}\right)-\tfrac{1}{3}(3R-d)\sqrt{d(2R-d)}\right].$$

But $d=2(R-r)$.

$$\therefore\ S=\frac{\sqrt{[h^2+(R-r)^2]}}{R-r}\left[\frac{\pi R^2}{2}-R^2\sin^{-1}\left(\frac{2r-R}{R}\right)-\tfrac{2}{3}(R+2r)\sqrt{r(R-r)}\right].$$

When $m^2<n^2$.

$$n(a+mc)\sqrt{(1+n^2)}\int_0^{y'}\frac{(c-y)dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=\frac{n(a+mc)\sqrt{(1+n^2)}}{\sqrt{[(n^2-m^2)^3]}}\int_{u_2}^{u_1}\frac{(cm^2+am-u)du}{\sqrt{[u^2-(mnc+an)^2]}}$$

$$=\frac{n(a+mc)\sqrt{(1+n^2)}}{\sqrt{[(n^2-m^2)^3]}}\sqrt{(n^2c+am)^2-(mnc+an)^2}$$

$$-\frac{nm(a+mc)^2\sqrt{(1+n^2)}}{\sqrt{[(n^2-m^2)^3]}}\log\left(\frac{n^2c+am+\sqrt{[(n^2c+am)^2-(mnc+an)^2]}}{mnc+an}\right).$$

$$\therefore\ S=n\sqrt{(1+n^2)}\left[\frac{\pi c^2}{2}-c^2\sin^{-1}\left(\frac{a}{nc}\right)\right.$$

$$-\frac{(a+mc)}{\sqrt{[(n^2-m^2)^3]}}\sqrt{(n^2c+am)^2-(mnc+an)^2}$$

$$\left.+\frac{m(a+mc)^2}{\sqrt{[(n^2-m^2)^3]}}\log\left(\frac{n^2c+am+\sqrt{[(n^2c+am)^2-(mnc+an)^2]}}{mnc+an}\right)\right].$$

$$\therefore S = \frac{\sqrt{[h^2+(R-r)^2]}}{R-r}\left[\frac{\pi R^2}{2} - R^2 \sin^{-1}\left(\frac{R-d}{R}\right)\right.$$

$$-\frac{rd(R-r)}{\sqrt{[d(2R-d-2r)^3]}}\sqrt{[(2R-r-d)^2-r^2]}$$

$$\left.+\frac{r^2d(r-R+d)}{\sqrt{[d(2R-d-2r)^3]}}\log\left(\frac{2R-r-d+\sqrt{[(2R-r-d)^2-r^2]}}{r}\right)\right].$$

For volume,

$$V = 2\int_0^{y'}\int_{x_2}^{x_1}\sqrt{[n^2(c-y)^2-x^2]}\,dy\,dx$$

$$=\int_0^{y'}\left[\tfrac{1}{2}\pi\, n^2(c-y)^2 - n^2(c-y)^2\sin^{-1}\left(\frac{a+my}{n(c-y)}\right)\right.$$

$$\left.-(a+my)\sqrt{[n^2(c-y)^2-(a+my)^2]}\right]dy = \tfrac{1}{6}\pi n^2c^3 - \tfrac{1}{3}n^2c^3\sin^{-1}\left(\frac{a}{nc}\right)$$

$$-\tfrac{1}{3}n^2(a+mc)\int_0^{y'}\frac{(c-y)^2\,dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$-\int_0^{y'}(a+my)\sqrt{[n^2(c-y)^2-(a+my)^2]}\,dy.$$

When $m^2>n^2$,

$$\tfrac{1}{3}n^2(a+mc)\int_0^{y'}\frac{(c-y)^2\,dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=\frac{n^2(a+mc)}{3\sqrt{[(m^2-n^2)^5]}}\int_{u_2}^{u_1}\frac{(cm^2+am-u)^2\,du}{\sqrt{[(an+mcn)^2-u^2]}}$$

$$=\frac{n^2(a+mc)}{6\sqrt{[(m^2-n^2)^5]}}\left[\frac{\pi(a+mc)^2(2m^2+n^2)}{2}\right.$$

$$-(a+mc)^2(2m^2+n^2)\sin^{-1}\left(\frac{n^2c+am}{an+nmc}\right)$$

$$\left.+(n^2c-4cm^2-3am)\sqrt{[(an+nmc)^2-(am+n^2c)^2]}\right].$$

$$\int_0^{y'}(a+my)\sqrt{[n^2(c-y)^2-(a+my)^2]}\,dy$$

$$=\frac{1}{\sqrt{[(m^2-n^2)^5]}}\int_{u_2}^{u_1}(mn-an^2-n^2cm)\sqrt{[(an+ncm)^2-u^2]}\,du$$

$$=\frac{1}{\sqrt{[(m^2-n^2)^5]}}\Big[\tfrac{1}{3}m[(an+cmn)^2-(n^2c+am)]^{\frac{3}{2}}$$

$$+\frac{n^4(a+cm)^3}{2}\sin^{-1}\left(\frac{n^2c+am}{an+nmc}\right)-\frac{\pi n^4(a+cm)^2}{4}$$

$$+\frac{n^2(a+mc)(am+n^2c)}{2}\sqrt{[(an+nmc)^2-(am+n^2c)^2]}\Big].$$

$$\therefore\ V=\tfrac{1}{6}\pi n^2c^3-\tfrac{1}{3}n^2c^3\sin^{-1}(a/nc)-\frac{\pi n^2(a+cm)^3}{6\sqrt{[(m^2-n^2)^3]}}$$

$$+\frac{n^2(a+cm)^3}{3\sqrt{[(m^2-n^2)^3]}}\sin^{-1}\left(\frac{am+n^2c}{an+cmn}\right)$$

$$+\frac{2acn^2+a^2m+n^2c^2m}{3\sqrt{[(m^2-n^2)^3]}}\sqrt{[(an+cnm)^2-(am+n^2c)^2]}.$$

$$\therefore\ V=\frac{R^3h}{3(R-r)}\left[\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{R-d}{R}\right)\right]$$

$$-\frac{hr^3d}{3(R-r)\sqrt{[d(d+2r-2R)^3]}}\left[\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{2R-r-d}{r}\right)\right]$$

$$+\frac{h[2Rd(R-d)+d^2(r-R+d)]}{3\sqrt{[d(d+2r-2R)^3]}}\sqrt{[r^2-(2R-r-d)^2]}.$$

Let $d=2R$. $\therefore\ V=\dfrac{\pi Rh}{3(R-r)}[R^2-r\sqrt{(Rr)}]$.

When $m^2=n^2$, $y'=[(nc-a)/2n]$.

$$\tfrac{1}{3}n^2(a+mc)\int_0^{y'}\frac{(c-y)dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=\tfrac{1}{3}n^2\sqrt{[a+nc]}\int_0^{y'}\frac{(c-y)^2dy}{\sqrt{[nc-a-2ny]}}=\frac{\sqrt{(n^2c^2-a^2)}(7n^2c^2+6anc+2a^2)}{45n}.$$

$$\int_0^{y'}(a+my)\sqrt{[n^2(c-y)^2-(a+my)^2]}dy$$

$$=\sqrt{[nc+a]}\int_0^{y'}(a+ny)\sqrt{[nc-a^2-2ny]}dy=\frac{(4a+nc)(nc-a)\sqrt{(n^2c^2-a^2)}}{15n}.$$

$$\therefore\ V=\tfrac{1}{6}\pi n^2c^3-\tfrac{1}{3}n^2c^3\sin^{-1}(a/nc)-\frac{(2n^2c^2+3anc-2a^2)\sqrt{(n^2c^2-a^2)}}{9n}.$$

$$\therefore\ V=\frac{R^3h}{3(R-r)}\left[\tfrac{1}{9}\pi-\sin^{-1}\left(\frac{R-d}{R}\right)\right]-\frac{h(3R^2+Rd-2d^2}{9(R-r)}\sqrt{[d(2R-d)]}.$$

But $d=2R-2r$.

$$\therefore\ V=\frac{h}{3(R-r)}\left[\frac{\pi R^3}{2}-R^3\sin^{-1}\left(\frac{2r-R}{R}\right)+\frac{2(3R^2-14Rr+8r^2)}{3}\sqrt{[r(R-r)]}\right].$$

When $m^2<n^2$,

$$\tfrac{1}{3}n^2(a+mc)\int_0^{y'}\frac{(c-y)^2\,dy}{\sqrt{[n^2(c-y)^2-(a+my)^2]}}$$

$$=-\frac{n^2(a+mc)}{3\sqrt{[(n^2-m^2)^5]}}\int_{u_2}^{u_1}\frac{(cm^2+am-u)^2\,du}{\sqrt{[u^2-(an+mcn)^2]}}$$

$$=-\frac{n^2(a+mc)}{6\sqrt{[(n^2-m^2)^5]}}\Big[(4cm^2+3am-n^2c)\sqrt{[(n^2c+am)^2-(an+mnc)^2]}$$

$$-(2m^2+n^2)(a+mc)^2\log\left(\frac{n^2c+am+\sqrt{[(n^2c+am)^2-(an+mnc)^2]}}{an+mnc}\right)\Big].$$

$$\int_0^{y}(a+my)\sqrt{[n^2(c-y)^2-(a+my)^2]}\,dy$$

$$=\frac{1}{\sqrt{[(n^2-m^2)^5]}}\int_{u_2}^{u_1}(mu-an^2-n^2cm)\sqrt{[u^2-(an+mnc)^2]}\,du$$

$$=\frac{1}{\sqrt{[(n^2-m^2)^5]}}\left[\frac{n^2(a+mc)(am+n^2c)}{2}\right]\sqrt{[(n^2c+am)^2-(an+nmc)^2]}$$

$$-\tfrac{1}{3}m[(n^2c+am)^2-(an+nmc)^2]^{\frac{3}{2}}$$

$$-\frac{n^4(a+cm)^3}{2}\log\left(\frac{n^2c+am+\sqrt{[(n^2c+am)^2-(an+nmc)^2]}}{an+nmc}\right).$$

$$\therefore\ V=\tfrac{1}{6}\pi n^2c^3-\tfrac{1}{3}n^2c^3\sin^{-1}\left(\frac{a}{nc}\right)$$

$$+\frac{n^2(a+mc)^3}{3\sqrt{[(n^2-m^2)^3]}}\log\left(\frac{n^2c+am+\sqrt{[(n^2c+am)^2-(an+nmc)^2]}}{an+nmc}\right)$$

$$-\frac{2n^2ac+am^2+n^2c^2m}{3\sqrt{[(n^2-m^2)^3]}}\sqrt{[(n^2c+am)^2-(an+nmc)^2]}.$$

$$\therefore\ V=\frac{R^3h}{3(R+r)}\left[\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{R-d}{R}\right)\right]$$

$$+\frac{hr^3d}{3(R-r)\sqrt{[d(2R-d-2r)^3]}}\log\left(\frac{2R-r-d+\sqrt{[(2R-r-d)^2-r^2]}}{r}\right)$$

$$-\frac{h[2Rd(R-d)+d^2(r-R+d)]}{3\sqrt{[d(2R-d-2r)^3]}}\sqrt{[(2R-r-d)^2-r^2}.$$

Let $m=0$.

$$\therefore\ V=\tfrac{1}{3}n^2c^3\left[\tfrac{1}{2}\pi-\sin^{-1}\left(\frac{a}{nc}\right)\right]$$

$$+\frac{a^3}{3n}\log\left(\frac{nc+\sqrt{[n^2c^2-a^2]}}{a}\right)-\tfrac{2}{3}ac\sqrt{[n^2c^2-a^2]}$$

$$=\tfrac{1}{3}c\left[\frac{\pi R^2}{2}-\sin^{-1}\left(\frac{a}{R}\right)+a^3\log\left(\frac{R+\sqrt{[R^2-a^2]}}{a}\right)-2a\sqrt{[R^2-a^2]}\right]$$

$$=\frac{Rh}{3(R-r)}\left[\frac{\pi R^2}{2}-R^2\sin^{-1}(r/R)+\frac{Rr^3}{R-r}\log\left(\frac{R+\sqrt{[R^2-r^2]}}{r}\right)-2r\sqrt{[R^2+r^2]}\right]$$

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

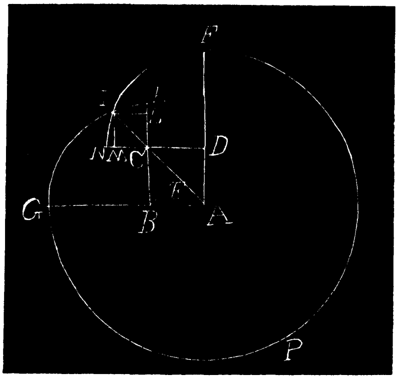
93. Proposed by **RAYMOND D. SMITH**, Tiffin, Ohio.

A barn 20 feet square is standing in a pasture, and a horse is tied to one corner of it with a rope 50 feet long. Over how much land can he graze ?

I. Solution by **B. F. FINKEL**, M. Sc., M. A., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let $ABCD$ be the barn, side $AB=AD=20$ feet; A the corner to which the horse is tied; and $AF=AG=50$ feet, the length of the rope.

Then $DI=BI=30$ feet; $AC=DB=20\sqrt{2}$ feet; $EI=\sqrt{[DI^2-DE^2]}=\sqrt{[30^2-(10\sqrt{2})^2]}$ feet$+10\sqrt{7}$ feet; $CI=EI-EC=10\sqrt{7}$ feet$-10\sqrt{2}$ feet$=10(\sqrt{7}-\sqrt{2})$ feet; $CL=CM=\sqrt{[CI^2/2]}=\frac{1}{2}CI\sqrt{2}=5(\sqrt{14}-2)$ feet; $KL=CK-CL=10$ feet$-5(\sqrt{14}-2)$ feet$=5(4-\sqrt{14})$ feet; and chord $KI=$chord $IN=\sqrt{[KL^2+IL^2]}$.



$\sqrt{[25(4-\sqrt{14})^2+25^2(\sqrt{14}-2)^2]}$ feet $=10\sqrt{[3(4-\sqrt{14}]}$ feet.

2 arc $IK=\frac{1}{3}(8$ chord $KI-2IL$*
$=\frac{1}{3}\{80\sqrt{[3(4-\sqrt{14})]}-20(\sqrt{7}-\sqrt{2})\}$ feet$=\frac{20}{3}\{4\sqrt{[3(4-\sqrt{14})]}-(\sqrt{7}-\sqrt{2})]$ feet.

The area over which the horse can graze$=FAGPF+$sector $FDI+$sector $IBG+$triangle $DCI+$triangle $BCI=FAGPF+2$ sector $FDI+2$ triangle $DCI=FAGPF+2$(quadrant FDN—sector $IDN)+2$ triangle DCI.

But area of $FAGPF=\frac{3}{4}\pi AF^2=1875\pi$;

*See *Williamson's Differential Calculus*, pages 64-65, for a proof of this rule. The discovery of this important approximation is due to Huygens. The length of an arc of 30° on a circle of radius 100,000 differs from the true value, assuming $\pi=3.141592$, by about 2 inches. The formula is arc$=\frac{1}{3}(8B-A)$ when B is the chord of half the arc and A is chord ofthe arc.

area of quadrant $FDN=\frac{1}{4}\pi DF^2=225\pi$; and area of sector $IDN : 2\pi DF^2 :: 2\pi DF$: arc IN, or area of sector $IDN=\frac{1}{2}DF\times$ arc $IN=15\{\frac{10}{3}[4\sqrt{3[(4-\sqrt{14})]}-\sqrt{7}+\sqrt{2}]\}$ square feet$=50\{4\sqrt{[3(4-\sqrt{14})}-\sqrt{7}+\sqrt{2}\}$ square feet.

$\therefore$ Area of sector $FDI=225\pi-50\{4\sqrt{[3(4-\sqrt{14})]}-\sqrt{7}+\sqrt{2}\}$.

Area of triangle $DCI=\frac{1}{2}DC\times IM=10\times5(\sqrt{14}-2)=50(\sqrt{14}-2)$ square feet. $\therefore$ The total area over which the horse can graze$=1875\pi+2(225\pi-50\{4\sqrt{[3(4-\sqrt{14})]}-\sqrt{7}+\sqrt{2}\})+2[50(\sqrt{14}-2)]=1875\pi+450\pi-1004\sqrt{[3(4-\sqrt{14})]}-\sqrt{7}+\sqrt{2}+100(\sqrt{14}-2)=2375\pi+100\sqrt{14}-2-4\sqrt{[3(4-\sqrt{14})]}+\sqrt{7}-\sqrt{2}=7249.378$ square feet.

II. Solution by G. I. HOPKINS, A. M., Professor of Mathematics and Physics, High School, Manchester, N. H.; J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.; and P. S. BERG, Principal of School, Larimore, N. D.

Let $ABCD$ represent the barn and A the corner to which the horse is tied. Then $FA=AG=50$ feet, and $DF=ID=BI=GB=30$. The area over which he can graze is divided into four parts, viz.: the three-quarters of a circle $AFPGA$, the two sectors GBI and IDF, and the quadrilateral $IBCD$.

$BD=\sqrt{(20^2+20^2)}=20\sqrt{2}$. $\therefore$ $ED=10\sqrt{2}$.

$\therefore$ $CE=\sqrt{[30^2-(10\sqrt{2})^2]}=10\sqrt{7}$.

Area $IBCD$=area $IBD-$area BCD.

$\therefore$ Area $IBCD=10\sqrt{7}\times10\sqrt{2}-200=100\sqrt{14}-200$.

$\therefore$ Area $IBCD=174.1657$ square feet.

$\cos\angle IDE=(10\sqrt{2})/30=.4714$.

$\therefore$ $\angle IDE=61^\circ\ 52'\ 30''$ and $\angle BDA=45^\circ$.

$\therefore$ $\angle IDA=106^\circ\ 52'\ 30''$. $\therefore$ $\angle IDF=73^\circ\ 7'\ 30''$.

Sectors GIC and IDF are equal. $\therefore$ $2\angle 73^\circ\ 7'\ 30''=146^\circ\ 15'=146\frac{1}{4}^\circ$.

Area of circle whose radius is $ID=30^2\pi=900\pi$. $\therefore$ The areas of the two sectors GBI and $IDF=(146\frac{1}{4}/360)\times900\pi=365\frac{5}{8}\pi=365.625\pi$.

Area of $GAFPG=(3\cdot50^2\pi)/4=1875\pi$. $(365.625+1875)\pi=2240.625\pi$.

$\therefore$ Area $GAFPG=7039.1475$ square feet.

$\therefore$ Entire area$=174.1657+7039.1475=7213.3132$ square feet$=26.495$ square rods.

This problem was also solved by G. B. M. Zerr who got as an answer 7291.9868 square feet; J. Scheffer, his answer being 6889.414 square feet; Fremont Crane, his result being 6351.785 square feet; and B. F. Sine, his result being 7233.292 square feet. Cooper D. Schmitt did not solve it, but referred to a previous solution in the MONTHLY.

94. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pa.

What rate of income do I realize by purchasing United States 4% bonds at 105 if I sell them in six years at 104?

Solution by CHARLES C. CROSS, Libertytown, Md.; FREMONT CRANE, Sand Coulee, Mont.; HON. JOSIAH H. DRUMMOND, Portland, Me.; and G. B. M. ZERR, Pottstown, Pa.

$.04\times6=24$.

$1.04+.24=1.28$, amount realized on bond.

$1.28-1.05=.23$, amount gained in six years.

$.23 \div 6 = .03\frac{5}{6}$, amount gained in one year.

$.03\frac{5}{6} \div 1.05 = 3\frac{4}{6}\frac{1}{3}\%$ = rate of income.

Also solved by *P. S. BERG.*

95. Proposed by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military College, Washington, Miss.

A man sold a house for \$7,500 and gained a certain per cent. on the cost. If the cost had been $16\frac{2}{3}\%$ less, his gain would have been 25% greater. Find the cost of the house.

I. Solution by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.; and M. A. GRUBER, A. M., War Department, Washington, D. C.

The increase in per cent. gained is made up from two sources: (a) additional *gain* caused by decrease in cost, and (b) original compared with supposed cost instead of actual cost.

$16\frac{2}{3}\%$ of cost $\div$(cost $-\frac{1}{6}$ cost) $=\frac{1}{5}$, or 20% of supposed cost.

Hence 25%$-$20%, or 5%, must be due to changing the *base*, and (gain $\div\frac{5}{6}$ cost)$-$(gain $\div$ cost)$=$5%, or $\frac{1}{20}$, or gain $\div 5\times$ cost $=\frac{1}{20}$, and gain $=\frac{1}{4}$ cost.

Hence $\frac{5}{4}$ cost $=$ \$7,500, and cost $=$ \$6,000.

Solution No. 2. The *rate* on $\frac{5}{6}$ cost $=\frac{6}{5}$ of rate on cost.

$\frac{1}{6}$ of cost $\div\frac{5}{6}$ cost $=\frac{1}{5}$, or 20%.

$\therefore$ $\frac{6}{5}$ of rate on cost $+$ 20% $=$ rate on cost $+$ 25%, or $\frac{1}{5}$ rate $=$ 5%, and rate $=$ 25%.

Hence \$7,500 $=$ 125% of cost, and cost $=$ \$6,000.

Solution No. 3. Let $6x$ = cost. Then \$7,500 $-6x$ = gain.

\$7,500 $-5x$ = supposed gain.

$$\frac{7500-5x}{5x}=\frac{7500-6x}{6x}+\tfrac{1}{4}, \text{ whence } x=\$1,000,\ 6x=\$6,000.$$

II. Solution by B. F. FINKEL, M. Sc., M. A., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

1. *100* per cent. = selling price.
2. 100 per cent. = actual cost.
3. *100* per cent. $-$ 100 per cent. = gain.
4. 100 per cent. $-$ $16\frac{2}{3}$ per cent. $=83\frac{1}{3}$ per cent. = conditional cost.
5. *100* per cent. $-$ $83\frac{1}{3}$ per cent. = conditional gain.
6. $83\frac{1}{3}$ per cent. = 100 per cent. of itself.
7. 1 per cent. $=1/83\frac{1}{3}$ of 100 per cent. $=1\frac{1}{5}$ per cent.
8. *100* per cent. $-83\frac{1}{3}$ per cent. $=$ (*100* $-83\frac{1}{3}$) times $1\frac{1}{5}$ per cent. $=\frac{6}{5}\times$ *100* per cent. $-$ 100 per cent. = conditional gain per cent.
9. $\therefore$ $\frac{6}{5}\times$ *100* per cent. $-$ 100 per cent. $-$ (*100* per cent. $-$ 100 per cent.) $=$ $\frac{1}{5}\times$ *100* per cent. = difference.
10. 25 per cent. = difference.
11. $\therefore$ $\frac{1}{5}\times$ *100* per cent. = 25 per cent.
12. *100* per cent. $=5\times25$ per cent, $=125$ per cent. = selling price in terms of actual cost.
13. $\therefore$ 125 per cent. = \$7,500.

14. 1 per cent. $=\frac{1}{125}$ of \$7,500 = \$60.

15. 100 per cent. = \$6000 = the cost of the house.

Also solved by *COOPER D. SCHMITT*, *F. R. HONEY*, *M. E. GRABER*, *EDW. R. ROBBINS*, *B. F. SINE*, *J. SCHEFFER*, *J. F. TRAVIS*, *EARL R. GIBSON*, *FREMONT CRANE*, *HENRY HEATON*, *J. H. DRUMMOND*, and *G. B. M. ZERR*.

Solved in a similar manner, though more briefly, by Walter H. Drane. Professor Drane has some doubts as to whether the above is strictly an arithmetical solution and asks the question, What is the exact difference between an arithmetical and an algebraic solution? This question I have answered in the last number of the MONTHLY. The above is not an arithmetical solution at all. Professor Ellwood's last solution is algebraic. My solution of problem 93, aside from the principles borrowed from geometry, is purely arithmetical, because all the operations are performed at once upon certain numbers and the results of these operations summed give the final result. The selling price is printed in italics for sake of distinction. [ED. F.]

ALGEBRA.

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83. Proposed by J. MARCUS BOORMAN, Consultative Mechanician and Counselor at Law, Woodmere, Long Island, New York.

Solve $x^2+y=8$......(I) ; $y^2+x=60$......(II), true to four decimals.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

(I) in (II) gives $x^4-16x^2+x=-4$.

$\therefore x=-4,\ y=-8.\quad \therefore x=3.935432,\ y=-7.487627$;

$x=.537401,\ y=7.711199$;

$x=-.472833,\ y=7.776428$.

Solved similarly by *J. H. DRUMMOND*, and *EDW. R. ROBBINS*.

II. Solution by the PROPOSER.

[N. B. Change $\pm_1$......$\mp_1$ *only* in unison,—thus $S={}_1\pm(x+x_1)$ to $\Sigma={}_1\pm(y+y_1)$; $S_1=(x_1+x_{11})$ to $\Sigma_1=(y_1+y_{11})$; $S_a=(x+x_{11})$ to $\Sigma_a=\pm_1(y+y_{11})$, 12 roots.]

The "*Empirical*" gives (E) ....

$S^4-2[S\pm_1\sqrt{(8^2-60)}]S^2+Id^2=Id^2\pm_1\frac{1}{4}f\sqrt{[1/(8^2-60)]}$. ["$f$"=1, factor of x in (I) ; S=sum $x+x_1$; Σ=*co*-sum $(y+y_1)$, etc. *Analyst* for 1882, IX, page 122].

$f=f^2=1$..... $\therefore$

(III)..... $S^2=10\pm\sqrt{100\frac{1}{8}}=10\pm10.0062484$.......$\therefore S=\pm4.472834,2+$,

(IV) ...... $S_1^2=6\pm\sqrt{35\frac{7}{8}}=6\pm\ 5.9895761$. True $S=\ 4.472833,9+$.

Error .....0.000000,3.

By (IV) $S_1=\pm3.462596,5$,—its reciprocal is $\Sigma_1=0.288800,4+$,

True $S_1=3.462598,42$. Reciprocal of S is $\Sigma=0.223571,9+$,

Error..... 0.000001,9 $\therefore S_a=0.064567,6$; $\Sigma_a=15.487627,6+$.

[Thus reducing the "ir-reducible" case]. Q. V. D.

By $\Sigma=f(1/S)$, $f=1=S\Sigma=SS_1S_a$. *Any* S or Σ will give, by

(A)......$x=\pm_1\frac{1}{2}S\pm\sqrt{[8-\frac{1}{4}S^2\mp_1\frac{1}{2}f(1/S)]}$. ($f=f^2=1$ here.)

(B)......$y=\pm_1\frac{1}{2}\Sigma\pm\sqrt{[60-\frac{1}{4}\Sigma^2\mp_1\frac{1}{2}f^2(1/\Sigma)]}$, the whole *eight* roots, viz.:

$x=0.537401+$, $x_1=3.935432$, $x_{11}=-0.472833,9$, $x_a=-4$;

$y=7.711199+$, $y_1=-7.487627$, $y_{11}=7.776428,1$, $y_a=-8$.

Compare $x_1=3.935432$	331970	029801	953731,229+
$x_a=-4.$			
$S_a=\mp 0.064568$	668029	970198	946268,770$\mp$
$y_1=-7.487627$	639515	066744	071631,521+
$y_a=8.$			
$(\Sigma)_a=\pm 0.512372$	360484	933255	928368,478−
$x=0.537401$	577025	225760	614153,4040+
$x_2=-0.472833$	908995	255561	667884,633+
$S_{a_1}=-0.064568$	668029	970198	946268,770$\pm$
Also, $y=7.711199$	545010	300343	929222,6193+;
$y_2=7.775428$	094504	266400	142408,902$\pm$.

S, Σ, etc., is stated $\pm$ because the remaining two roots of x or y *so* make it. Note that $S_{a_1}+=-S_a$, and that the two determine the final figure 0 ! Also observe the curious recurrences 777 *twice*, 222,| 111, etc.

The "*Empirical*" is one of three novelties found in 1875-6. Des Cartes' $S^6-4aS^4+4bS^2-f^2=0$ . . . . . (C), *so stated* factors into $S^3\mp_1 2xS^2-2yS\pm_1 f=0$! Also to $S^3-2[a\pm_1 | (a^2-b)]S\mp_1 \eta| \surd[1/(a^2-b)](1/S)=0$ . . . . . .(D)!

Change $\eta|\surd[1/(a^2-b)](1/S)$ "*empirically*" into $\frac{1}{4}f|\surd[1(a^2-b)](1/S)$ gives (E) above. For, say twelve decimals *true correct* $\frac{1}{4}f$, etc., into $(\frac{1}{4}+)v$, etc., by $v+(\frac{1}{4}+)^2v^2[1/(a^2-b)](1/S^2)=f$.

NOTE. Josiah H. Drummond remarks of the Proposer's solution of No. 78: It seems to me that his reasoning is faulty, and his conclusion erroneous. His reasoning seems to be: If $x^2=36$ (1) it follows that $x^4=1296$ (2) and as (2) has *four* roots, therefore (1) has four roots. Again +2 and −2, *taken by themselves*, are precisely the same; the first is +2 or −2, and the second is precisely the same, save *in the order of writing or reading them*, which does not affect the mathematical result.''

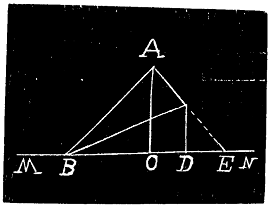
GEOMETRY.

91. Proposed by LEONARD E. DICKSON, Ph. D., Instructor in Mathematics in the University of California, Berkeley, Cal.

If a point A remain fixed while a point B moves along a given straight line, prove that the locus of the vertex C of the triangle ABC, similar to a given triangle and lying always on the same side of AB, is a straight line. Verify geometrically for the case in which the angles at A and C remain equal.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let MN be the fixed line, denote AO, which is constant, by h, and the coordinates OD by x and CD by y, and extend AC to E, denote $\angle AEB$ by E, and the constant angles of triangle ABC by A, B, C. We have



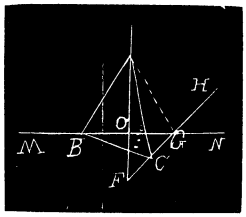
$$AB=\frac{h}{\sin(A+E)},\ BC=\frac{y}{\sin(C-E)};$$

$$\therefore \frac{h}{\sin(A+E)} : \frac{y}{\sin(C-E)}=\sin C : \sin A,$$

or, $\dfrac{h(\sin C\sin E-\cos C\sin E)}{y(\sin A\cos E+\cos A\sin E)}=\dfrac{\sin C}{\sin A}$; or $h(1-\cot C\tan E)=y(1+\cot A\tan E)$, or

$$h-y=(h\cot C+y\cot A)\tan E\text{; but }\tan E=(h-y)/x,$$

$\therefore$ $x=y\cot A+h\cot C$, or $y=x\tan A-h\cot C\tan A$, as the locus of C.



For $C=A$, we have $y=x\tan A-h$.

Make $OF=OA$, draw FH so as to make $\angle HGN=\angle A$, draw AG. $\angle HGN=OGF=\angle AGO=\angle A$.

Since $\angle BAC=\angle BGC$, $ABCG$ is concyclic.

$\therefore$ $\angle ACB=\angle AGB=\angle A$, $\therefore$ $\angle C=\angle A$, which unifies geometrically for the case in which the angles at A and C remain equal.

92. **Proprosed by JOSIAH H. DRUMMOND, LL. D., Counselor at Law, Portland, Me.**

Let $ABCD$ be a quadrilateral inscribed in a circle. Draw the diagonals AC and BD. Show that $AB.BC : DC.AD=BD : AC$. [From a note in *Young's Geometry*, edition of 1830.]

Solution by the PROPOSER, and J. SCHEFFER, Hagerstown, Md.

Let F be the intersection of the diagonals.

Then $AB : BF :: CD : CF$,

or $AB : CD :: BF : CF$,

and $BC : AD :: CF : DF$,

Hence $AB.BC : AD.CD :: BF : DF$,

(I) and $AB.BC+AD.CD : AD.CD : BF+DF(=BD) : DF$.

In like manner it is shown

(II) that $AB.AD+BC.CD : BC.DC :: AC : CF$.

But $AD : DF :: BC : CF$,

or $AD.CD : DF :: BC.CD : CF$.

Combining these with (I) and (II), we have

$AB.BC+AD.CD : AB.AD+BC.CD :: BD : AC$. Q. E. D.

Also solved by *B. F. SINE*, *CHAS. C. CROSS*, *WALTER H. DRANE*, and *G. B. M. ZERR*.

93. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

While surveying in a level field I notice a mountain behind a hill. Wishing to know the height of each I take the angles of elevation of the tops of both and find them to $\beta=45^\circ$, $\delta=40^\circ$. I then measure a straight line $a=400$ feet, and find the angles of elevation of the tops to be $\gamma=42^\circ$, $\mu=38^\circ$. After measuring $b=300$ feet more in the same straight line I find the elevations to be $\lambda=40^\circ$, $\nu=36^\circ$. Find the height of each.

Solution by the PROPOSER.

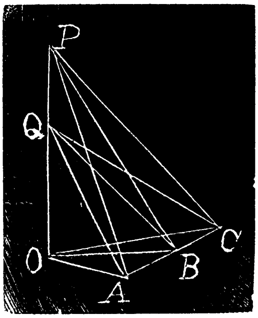
Let $AB=400$ feet$=a$, $BC=300$ feet$=b$, $OP=x$, $QR=y$.

$\angle OAP=45°=\beta$, $\angle OAQ=40°=\delta$, $\angle OBP=42°=\gamma$, $\angle OBQ=38°=\mu$, $\angle OCP=40°=\lambda$, $\angle OCQ=36°=\nu$.

$OA=m$, $OB=n$, $OC=p$, $\angle OCA=\theta$.

$\therefore$ $m=y\cot\delta=x\cot\beta$, $n=y\cot\mu=x\cot\gamma$.

$p=y\cot\nu=x\cot\lambda$.



Also from triangles OCA and OCB,

$m^2=p^2+(a+b)^2-2p(a+b)\cos\theta$ ....(1).

$n^2=p^2+b^2-2pb\cos\theta$ ....(2).

The values of m, n, p in (1) and (2) and eliminating $\cos\theta$, we get

$$x=\sqrt{\frac{ab(a+b)}{b\cot^2\beta+a\cot^2\lambda-(a+b)\cot^2\gamma}} \qquad y=\sqrt{\frac{ab(a+b)}{b\cot^2\delta+a\cot^2\nu-(a+b)\cot^2\mu}}$$

Substituting we get $y=1505.183$ feet, $x=4232.505$ feet.

Also solved by *ALOIS F. KAVORIK*, and *CHAS. C. CROSS*.

CALCULUS.

71. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Form the differential equation of the third order, of which $y=c_1e^{2x}+c_2e^{-3x}+c_3e^x$ is the complete primitive.

III. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, O.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; C. W. M. BLACK, A.M., Professor of Mathematics, Wesleyan Academy, Wilbraham, Mass.; G. B. M. ZERR, A. M., Ph. D., The Russell College, Lebanon, Va.; and the PROPOSER.

$y=c_1e^{2x}+c_2e^{-3x}+c_3e^x$ .....(1), $dy/dx=2c_1e^{2x}-3c_2e^{-3x}+c_3e^x$........(2); $d^2y/dx^2=4c_1e^{2x}+9c_2e^{-3x}+c_3e^x$....(3). $d^3y/dx^3=8c_1e^{2x}-27c_2e^{-3x}+c_3e^x$....(4).

(2)−(1) gives $dy/dx-y=c_1e^{2x}-4c_2e^{-3x}$ ....(5),

(4)−(1) gives $d^3y/dx^3-y=7c_1e^{2x}-28c_2e^{-3x}$ ....(6),

(6)−7(5) gives $d^3y/dx^3-7(dy/dx)+6y=0$.

IV. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

The equation reduces itself to what is the equation the three roots of which are 1, 2, −3. This equation is $z^3-7z+6=0$; consequently the complete primitive is $d^3y/dx^3-7(dy/dx)+6y=0$.

72. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

A man has a park in the form of a parabolic segment cut off by a chord making an angle $\frac{1}{4}\pi$ with the axis. Within the park is a right angled triangular flower plat with one vertex at the center of gravity of the segment, the other vertex at the lower extremity of the chord, and the right angle on the diameter bisecting the chord. The park contains 30

acres, and the perimeter of triangle in linear measure equals the area in square measure. Find the length of the chord, the latus-rectum of the parabola, and the dimensions of the triangle.

Solution by the PROPOSER.

Let $PQ=2y$, $BC=x$, $\angle EBC=\theta=\frac{1}{4}\pi$. $\therefore$ $y^2=4ax/\sin^2\theta=8ax$.

$\therefore$ Area$=2\sin\theta\int_0^x ydx=4\sqrt{a}\int_0^x x^{\frac{1}{2}}\,dx=(\frac{8}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}=\frac{1}{3}(2xy\sqrt{2})$.

$\therefore$ $\frac{1}{3}(2\sqrt{2xy})=4800$ square rods, or $xy=3600\sqrt{2}$ ....(1).

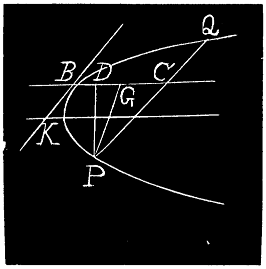
If G be the center of gravity, then $BG=3x/5$.

$PD=\frac{1}{2}y\sqrt{2}$, $GD=GC-DC=\frac{2}{5}x-\frac{1}{2}y\sqrt{2}=\frac{1}{10}(4x-5y\sqrt{2})$.

$PG=\sqrt{(PD^2+GD^2)}=\frac{1}{10}\sqrt{(100y^2+16x^2-40xy\sqrt{2})}$.

$\therefore$ $PG=\frac{1}{10}\sqrt{(100y^2+16x^2-2800)}$,

$\frac{1}{2}PD.DG=\frac{1}{40}(4xy\sqrt{2}-10y^2)\frac{1}{4}(2880-y^2)$.



$\therefore$ $\frac{1}{10}\sqrt{(100y^2+16x^2-2800)}+\frac{2}{5}x=\frac{1}{4}(2880-y^2)$.

$\therefore$ $2\sqrt{(100y^2+16x^2-2800)}=14400-5y^2-8x$.

$\therefore$ $25y^4-144400y^2+80xy^2-230400x$

$+208512000=0$ ....(2).

(1) in (2) gives

$y^5-5776y^3+11520\sqrt{2}y^2+8340480y=33177600\sqrt{2}$,

$y^5=5776y^3+16291.740672y^2+8340480y=46920213.13536$.

$\therefore$ $y=5.6903$ *nearly*, $2y=11.3806$, length of chord.

$x=894.7101$, $PD=4.0236$, $DG=353.3604$, $PG=353.8833$, area $PDG=711.7673$, $4a=.0181=$latus rectum.

Solved with different results by *C. W. M. BLACK*, and *J. SCHEFFER*.

NOTE on solution of Problem 69, Calculus, April number, page 111: "It seems to me that this solution does not solve this question. The fence prevents the horse from grazing on the ground within it; then, the rope must extend from one end of the major axis around, *outside of the fence*, to the other end, and is twice as long as that half of the fence. Hence the horse may graze around to the end of the minor axis on the other side of the field. The horse, starting from there and keeping the rope tight, will describe a curve as the rope unwinds from the fence, until he arrives at a point opposite the other end of the minor axis, being then half way around: proceeding, he reaches the other end of the minor axis (his starting point) and describes the other half of the curve. *Josiah H. Drummond*.

MECHANICS.

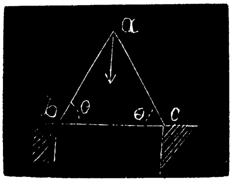
64. Proposed by FREDERIC R. HONEY, Ph. B., Instructor in Trinity College, New Haven, Conn.

Let the isosceles triangle abc, whose plane is vertical, and whose base bc is horizontal, and supported at each end b and c, represent three rods jointed at the points a, b, and c. Let any load L be suspended at the vertex a. It is required to find the value of the angle between the sides of the triangle and the base which shall make the sum of the weights of the rods a minimum, the length of the base bc being fixed.

I. Solution by the PROPOSER.

The rods ab and ac are in compression. Let C=number of pounds per square inch the material resists in compression. The rod bc is in tension. Let T=number of pounds per square inch the material resists in tension.

Let W=sum of weight of rods.



Let L=load.

Let w=weight per cubic inch of material employed.

Let θ=angle between the sides of the triangle and the base.

Length of rods $ab+ac=bc\times\sec\theta$.

Tension in rod $bc=\frac{1}{2}L\cot\theta$.

Compression in each of the rods ab and $ac=\frac{1}{2}L\operatorname{cosec}\theta$.

Number of square inches in section area of rod bc needed to resist the tension$=\frac{1}{2}L\cot\theta/T$.

Number of square inches in section area of each of the rods ab and $ac=\frac{1}{2}L\operatorname{cosec}\theta/C$.

Weight of rod bc=length$\times$section area$\times$weight of cubic inch of material $=bc\times(\frac{1}{2}L\cot\theta/T)\times w$.

Similarly, weight of rods $ab+ac=bc\sec\theta\times(\frac{1}{2}L\operatorname{cosec}\theta/C)\times w$.

And $W=bc\times(\frac{1}{2}L\cot\theta/T)\times w+bc\sec\theta\times(\frac{1}{2}L\operatorname{cosec}\theta/C)\times w$

$$=bc\times\tfrac{1}{2}Lw\left(\frac{\sec\theta\operatorname{cosec}\theta}{C}+\frac{\cot\theta}{T}\right).$$

Differentiating, $dW/d\theta=bc\times$

$$\tfrac{1}{2}Lw\left(\frac{\sec\theta\operatorname{cosec}\theta\tan\theta-\sec\theta\operatorname{cosec}\theta\cot\theta}{C}-\frac{\operatorname{cosec}^2\theta}{T}\right).$$

Putting $dW/d\theta=0$, we have

$$bc\times\tfrac{1}{2}Lw\left(\frac{\sec\theta\operatorname{cosec}\theta\tan\theta-\sec\theta\operatorname{cosec}\theta\cot\theta}{C}-\frac{\operatorname{cosec}^2\theta}{T}\right)=0.$$

Dividing by $bc\times\frac{1}{2}Lw\operatorname{cosec}\theta$, and transposing,

$$\frac{\sec\theta\tan\theta-\sec\theta\cot\theta}{C}=\frac{\operatorname{cosec}\theta}{T}.$$

Dividing by $\sec\theta$, $[(\tan\theta-\cot\theta)/C]=\cot\theta/T$.

$T\tan\theta-T\cot\theta=C\cot\theta$.

Dividing by $\cot\theta$, $T\tan^2\theta-T=C$, $\tan^2\theta=[(C+T)/T]$, $\tan\theta=\sqrt{[(C+T)/T]}$.

II. **Solution by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering in Agricultural and Mechanical College of Texas, College Station, Texas.**

The sum of the weights of the rods will be a minimum when their areas are a minimum, which will occur when the stresses are made a minimum. By resolution of forces we have for the sum of the three stresses in ab, bc and ac, calling the equal angles θ: sum of stresses equals

$$L\left(\frac{1}{\sin\theta}-\tfrac{1}{2}\frac{\cos\theta}{\sin\theta}\right).$$

Equating the first derivative to zero, we get, after reduction, $\cos\theta=\frac{1}{2}$.

Therefore $\theta=60^\circ$ and the triangle is equilateral.

65. **Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

The distance, parallel to the axis, from the mid-point of a chord to the arc of a parabola, is constant. Show that the center of gravity of all segments formed by the chord is an equal parabola.

I. **Solution by the PROPOSER.**

Let A be the vertex of the segment; (h, k) its coördinates; $5b$ the constant length from A to the center of the chord; θ the inclination of the chord to the axis; $y^2=4ax$, the equation to the parabola; (m, n) the coördinates of the center of gravity of the segment.

Then $h=m=a\cot^2\theta$, $n=k+3b=2a\cot\theta+3b$.

$\therefore$ $\cot\theta=[(n-3b)/2a]=\sqrt{(m/a)}$.

$\therefore$ $m/a=[(n-3b)/2a]^2$. Let $n=p+3b$.

$\therefore$ $p^2=4am$, an equal parabola.

II. **Solution by GEORGE LILLEY, Ph. D., Professor of Mathematics, University of Oregon, Eugene, Ore.**

Take the diameter through the mid-point and the tangent at its extremity for axes.

Then $y^2=4cx$, where θ is the angle between the axes and $c=a/\sin^2\theta$.

Since the diameter bisects the area of the segment,

$$x=\frac{\int_0^k 2x\sqrt{(cx)}dx}{\int_0^k 2\sqrt{(cx)}dx}=\tfrac{3}{5}k\ ;\ y=0,$$

where k is the distance from the arc to the mid-point.

But $x=a\cot^2\theta+\frac{3}{5}k$; $y=2a\cot\theta$, referred to the vertex and rectangular axes.

$\therefore$ $(y)^2=4a(x-\frac{3}{5}k)$ or $y^2=4a(x-\frac{3}{5}k)$, which is the equation of an equal parabola with its vertex on the axis at a distance $\frac{3}{5}k$ from the given one.

Also solved by *J. SCHEFFER*.

DIOPHANTINE ANALYSIS.

62. Proposed by JOHN M. ARNOLD, Crompton, R. I.

Find, if possible, four square numbers in arithmetical progression.

I. Solution in Imaginaries by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

I have not, thus far, succeeded in obtaining a solution in real numbers, but the following in imaginaries.

Let $(x-y)^2$, x^2+y^2, $(x+y)^2$, $(x+y)^2+2xy$ be the numbers.

Let $x=m^2-n^2$, $y=2mn$.

$\therefore (m^2-n^2-2mn)^2$, $(m^2+n^2)^2$, $(m^2-n^2+2mn)^2$, $(m^2-n^2+2mn)^2+4mn(m^2-n^2)$.

The last is a square when $m=n$; let $m=n+1$.

$\therefore (1-2n^2)^2$, $(2n^2+2n+1)^2$, $(2n^2+4n+1)^2$, $4n^4+24n^3+32n^2+12n+1$.

Assume, $4n^4+24n^3+32n^2+12n+1=(2n^2+6n+k)^2$; also $4n^4+24n^3+(32+a^2)n^2+(12+2ab)n+1+b^2=(an+b)^2$.

$\therefore a^2=4+4k$, $6+ab=6k$, $k^2=1+b^2$.

$\therefore k^3-8k^2+17k-10=0$. $\therefore k=1$ or 2 or 5.

$\therefore n=0$ or $-\frac{1}{2}\pm\frac{1}{2}\sqrt{(-11)}$ or -1.

$\therefore 25\pm12\sqrt{(-11)}$, 25, $25\mp12\sqrt{(-11)}$, $25\mp24\sqrt{(-11)}$, or $[6\pm\sqrt{(-11)}]^2$, $(5)^2$, $[6\mp\sqrt{(-11)}]^2$, $\{\sqrt{[(25+\sqrt{6961})/2]}\mp\sqrt{[(25-\sqrt{6961})/2]}\}^2$.

II. Comment by JOSIAH H. DRUMMOND, LL. D., Counselor at Law, Portland, Me.

Let $x^2-2xy+y^2$, x^2+y^2, $x^2+2xy+y^2$, $x^2+4xy+y^2$ be the four numbers; two of them being squares, we have to make $x^2+y^2=\square$ ......(1), and $x^2+4xy+y^2=\square$ ......(2). Let $x=my$, then from (1) $m^2+1=\square=$(say) $(p-m)^2$, from which $m=(p^2-1)/2p$. From (2), $m^2+4m+1=\square=$(say) $(pm-1)^2$. $m=(2p+4)/(p^2-1)=(p^2-1)/2p$, or $(p^2-1)^2=4p^2+8p$. Hence $4p^2+8p=\square$. Or $p^2+2p=\square=$(say) $q^2-2pq+p^2$, and $p=q^2/[2(q+1)]$. Then $4p^2+8p=[(q^2+2q)/(q+1)]^2$, and $p^2-1=(q^2+2q)/(q+1)$, and $p^2=(q^2+3q+1)/(q+1)$;

$$p=\sqrt{[(q^2+4q^2+4q+1)]}/(q+1).$$

The only methods, which I know, of making the numerator rational, give $q=0$, and $p=1$, and $m=0$.

Taking $p^2+2p=p^2q^2$ and proceeding in a similar manner as above, we get $p=[1/(q^2-1)]\sqrt{(q^4+4q^3-2q^2-4q+1)}$, and we get $q=0$, and $p=-1$ or $q=-1$ and $p=0$.

I have tried many other methods and all give $p=1$, or $p=-1$. While all this does not *demonstrate* that the question is impossible, I shall believe that it is so, until I see a solution.

63. Proposed by A. H. HOLMES, Brunswick, Me.

Given $x^3+y^3=20^3\times105489$, to find four positive *integral* values each for x and y.

Solution by the PROPOSER.

$x^3+y^3=20^3\times105489$ or $=843912000$. Take $x=1$, 2, etc., until we find $x=15$, $y=945$. Put $945=a$ and $15=b$. Then since there must be four values for each, we have. $x^3+y^3=a^3+b^3$, or $x^3-b^3=a^3-y^3$ ....(1).

Now suppose $x^3+b^3=a^3-y^3$. Let $x=a-u$ and $b=mu-y$.

$\therefore$ $a^3-3a^2u+3au^2-u^3+m^3u^3-3m^3a^2y+3muy^2-y^3=a^3-y^3$.

Let $m=a^2/y^2$ and we have $u=3ay^3/(a^3+y^3)$.

$\therefore$ $x=[a(a^3-2y^3)]/(a^3+y^3)$ and $b=[y(2a^3-y^3)]/(a^3+y^3)$.

Now to find x and y in $x^3-b^3=a^3-y^3$ it is evident one of the above values must be taken negatively, but it cannot be the value of b since the result would be at least one negative value.

$\therefore$ in (1) we have, $x=[y(2a^3-y^3)]/(a^3+y^3)$ and $b=[a(2y^3-a^3)]/(a^3+y^3)$. Whence we find $y=a\sqrt[3]{[(a+b)/(2a-b)]}=945\sqrt[3]{(\frac{960}{1875})}=945\sqrt[3]{(\frac{64}{125})}$.

$\therefore$ $y=756$, and we find $x=744$.

$\therefore$ $x=15$, 744, 756, and 945 ; $y=945$, 756, 744, and 15.

Also solved by *J. H. DRUMMOND*.

74. Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.

It is required to take from the proper *key* suitable material and hastily construct a "nest" of 10 or 15 prime, integral, rational trapeziums, each containing an area equal to the square root of the product of its four sides.

Solution by the PROPOSER.

If the business require great haste, write n, —two or more, convergents in the expansion of any number of the form of $\sqrt{(a^2+1)}$, say $\sqrt{2}$, $\sqrt{5}$, $\sqrt{10}$, $\sqrt{17}$, $\sqrt{26}$, $\sqrt{37}$, etc. Observe the number of trapeziums, $[n(n-1)]/2$ is always triangular.

$\sqrt{17}=\frac{4}{1}$, $\frac{33}{8}$: $4^2=\mathbf{16}$, $17\times1^2=\mathbf{17}$, $17\times8^2=\mathbf{1088}$, $33^2=\mathbf{1089}$.

$\sqrt{10}=\frac{3}{1}$, $\frac{19}{6}$, $\frac{117}{37}$: $3^2=9$, $10\times1^2=10$, $19^2=361$, $10\times6^2=360$, $117^2=13689$, $10\times37^2=13690$. 9, 10, 360, 361 ; 9, 10, 13689, 13690 ; 360, 361, 13689, 13690.

$\sqrt{5}=\frac{2}{1}$, $\frac{9}{4}$, $\frac{38}{17}$, $\frac{161}{72}$: 4, 5, 80, 81 ; 4, 5, 1444, 1445 ; 4, 5, 25920, 25921 ; 80, 81, 1444, 1445 ; 80, 81, 25920, 25921 ; 1444, 1445, 25920, 25921.

$\sqrt{2}=\frac{1}{1}$, $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, $\frac{41}{29}$: 1, 2, 8, 9 ; 1, 2, 49, 50 ; 1, 2, 288, 289 ; 1, 2, 1681, 1682 ; 8, 9, 49, 50 ; 8, 9, 288, 289 ; 8, 9, 1681, 1682 ; 49, 50, 288, 289 ; 49, 50, 1681, 1682 ; 288, 289, 1681, 1682.

Should it be desired to obtain several nests of this kind of rational trapeziums from a single series, take the convergents from the expansion of quantities of the form $\sqrt{(a^2+b)}$, where b is greater than 1.

$\sqrt{3}=\frac{1}{1}$, **2/1**, $\frac{5}{3}$, **7/7**, $\frac{19}{11}$, **26/15**. $\frac{71}{41}$, **97/56**, $\frac{265}{153}$: 3, 4, 48, 49 ; 3, 4, 675, 676 ; 3, 4, 9408, 9409 ; 48, 49, 675, 676 ; 48, 49, 9408, 9409 ; 675, 676, 9408, 9409 ; .....

1, 3, 25, 27 ; 1, 3, 361, 363 ; 1, 3, 5041, 5043 ; 1, 3, 70225, 70227 ; 27, 25, 361, 363 ; 27, 25, 5041, 5043 ; 27, 25, 70225, 70227 ; 361, 363, 5041, 5043 ; 361, 363, 70225, 70227 ; 5041, 5043, 70225, 70227.

$\sqrt{6}=\frac{2}{1}\ldots\ldots\frac{22}{9}\ldots\ldots\frac{218}{89}\ldots\ldots\frac{2158}{881}\ldots\ldots\frac{21362}{8721}\ldots\ldots\frac{211462}{86329}$

Here is material for 15 trapeziums of the kind required.

$\sqrt{6}\doteq\frac{5}{2}\ldots\ldots\frac{49}{20}\ldots\ldots\frac{485}{198}\ldots\ldots\frac{4801}{1960}\ldots\ldots\frac{47525}{19402}\ldots\ldots\frac{470449}{192060}\ldots\ldots\frac{4656965}{1901198}$.

These convergents furnish material for 21 trapeziums.

$\sqrt{7}=\frac{2}{1}, \frac{3}{1}. \frac{5}{2}, \frac{8}{3}, \frac{37}{14}, \frac{45}{17}, \frac{82}{31}, \frac{127}{48}, \frac{590}{223}, \frac{717}{271}, \frac{1307}{494}, \frac{2024}{765}. \frac{9403}{3554}, \frac{11427}{4319}, \frac{20830}{7873}, \frac{32257}{12192}, \frac{149858}{56641}, \frac{182115}{68833}$. In the nine odd convergents may be found the roots of 36 trapeziums. In the 2d, 6th, 10th, 14th, 18th, the roots of 10 trapeziums, and in the 4th, 8th, 12th, 16th, the seed-corn of 6 trapeziums of the kind desired.

Also solved by *G. B. M. ZERR.*

NOTE. Professor Zerr's solution of No. 60 is splendid, but he has left out a very difficult part of the solution, and has not referred us to the solution elsewhere. h, k, l, m, n, and p must be so taken that the sum of their fifth powers is a perfect fifth power, and I do not remember where I can find a solution of that question. *J. H. Drummond.*

MISCELLANEOUS.

59. II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics, Wesleyan Academy, Wilbraham, Mass.

[See problem and solution I, in May number, page 156.] Consider the rays of light in a plane parallel to the axis of cylinder. It is easily seen that they are all reflected from an element of the cylinder, parallel and in another plane which together with the given plane makes equal angles with the normal plane along the reflecting element. These planes of reflection, being all perpendicular to the base, will intersect in lines perpendicular to the base, and the locus of the ultimate intersections of adjacent planes will be a caustic cylinder whose elements are perpendicular to the base of the given cylinder. As the direction of the rays of light is still oblique the intersection of the caustic cylinder with the base will be a luminous curve. The elevation of the sun does not affect the positions of the planes of light, and so does not affect the form of the caustic cylinder or of the luminous curve.

The curve is the envelope of the projection upon the base of the successive planes of light. Let circle represent base of cylinder, AB projection of incident plane, BC of reflected plane, AB being parallel to axis of X (see figure in May number, page 156.)

$OB=r$, $\angle ABO=\angle CBO=\angle BOK=\alpha$, $BKX=2\alpha$. Equation to BK is $y=(x-OK)\tan2\alpha=(x-\frac{1}{2}r\sec\alpha)\tan2\alpha$.

Let $\tan\alpha=a$. Then $y=[x-\frac{1}{2}r\sqrt{(1+a^2)}][(2a)/(1-a^2)]$............(1).

Differentiating with reference to a, equating to zero, and clearing of fractions, we have

$$(a^2-1)a^2r+2x(1+a^2)^{\frac{3}{2}}-r(1+a^2)^2=0\,;$$

whence $x/r=(1+3a^2)/[2(1+a^2)^{\frac{3}{2}}]$. Substituting in (1),

$$y/r=\{(1+3a^2)/[2(1+a^2)^{\frac{3}{2}}]-[(1+a^2)^{\frac{1}{2}}/2]\}[(2a)/(1-a^2)]=a^3/(1+a^2)^{\frac{3}{2}}.$$

$$\therefore\ (y/r)^{\frac{1}{3}}=a/(1+a^2)^{\frac{1}{2}}\,;\ (y/r)^{\frac{1}{3}}=a^2/(1+a^2).$$

$a^2=(y/r)^{\frac{2}{3}}/[1-(y/r)^{\frac{2}{3}}]$; $\sqrt{(1+a^2)}=1/\sqrt{[1-(y/r)^{\frac{2}{3}}]}$ .... (2).

$$1-a^2=[1-2(y/r)^{\frac{2}{3}}]/[1-(y/r)^{\frac{2}{3}}].$$

$$a/(1-a^2=(y/r)^{\frac{1}{3}}\sqrt{[1-(y/r)^{\frac{2}{3}}]}/[1-2(y/r)^{\frac{2}{3}}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Substitute (2) and (3) in (1),

$$y=\left(x-\frac{r}{2\sqrt{[1-(y/r)^{\frac{2}{3}}]}}\right)\times\frac{2(y/r)^{\frac{1}{3}}\sqrt{[1-(y/r)^{\frac{2}{3}}]}}{1-2(y/r)^{\frac{2}{3}}};$$

$$x=\frac{y[1-2(y/r)^{\frac{2}{3}}]}{2(y/r)^{\frac{1}{3}}\sqrt{[1-(y/r)^{\frac{2}{3}}]}}+\frac{r}{2\sqrt{[1-(y/r)^{\frac{2}{3}}]}},$$

or $x=\frac{1}{2}(r^{\frac{2}{3}}+2y^{\frac{2}{3}})\sqrt{(r^{\frac{2}{3}}-y^{\frac{2}{3}})}$, the equation sought.

60. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

When the sun's declination is 23° 27′ 15″ North$=\delta$, in what latitude will it shine on the *north* side of buildings during the first half of the forenoon, and on the *south* side during the other half, and what will be the length of the day ?

I. Solution by the PROPOSER.

Let P be the north pole, PA a meridian, Z the zenith, HS the horizon, the sun rising at S, and describing the small circle arc SBA, during the forenoon, reaching A at apparent noon, ZH a prime vertical, bisecting the semi-diurnal arc SBA at B, and right-angled at Z, the declination$=\delta$, and latitude$=\lambda$, $\tan\delta=\alpha$, and $\tan\lambda=\chi$.

Then $PB=90^\circ-\delta$, $PZ=90^\circ-\lambda$, and $\angle ZPB=h=$the hour-angle at the pole, and measures the time of describing the arc BA on the south side of ZH.

$\cos h=\tan\delta\cot\lambda=\alpha\chi$.

The hour angle measuring $SBA=2h$, and $\cos 2h=-\tan\delta\tan\lambda=-\alpha\chi$. But $\cos 2h=(2\alpha^2/\chi^2)-1=-\alpha\chi$. $\therefore\ \chi^3-(\chi^2/\alpha)+2\alpha=0$. Solving this cubic, gives a positive root $\chi=2.1099$, and $\lambda=64^\circ\ 38'\ 30''$.

$\therefore\ h=156^\circ\ 16'\ 4''=10$ hours, 25 minutes, 4 seconds, and $2h=20$ hours, 50 minutes, 8 seconds, length of day.

II. Solution by JOHN M. ARNOLD, Crompton, R. I.

In the triangle ABC formed by the zenith, the pole, and the sun when crossing the prime vertical, $A=90^\circ$, $B=$the hour angle, $a=$sun's polar distance, and $c=$the co-latitude.

In the triangle $A'B'C'$ formed by the north point of the horizon, the pole, and the sun when rising, $A'=90^\circ$, $B'=$supplement of the hour angle at rising, $c'=$the latitude, and $a'=a=90^\circ-\delta=66^\circ\ 32'\ 45''$.

From trigonometry, $\cos B=\cot a\tan c$......(1) ; $\cos B'=\cot a'\tan c'$ .....(2) ; $c'+c=90^\circ$....(3). From the conditions of the problem, $B'=180^\circ-2B$....(4). From (2) and (3), $\cos B'=\cot a\cot c$. Eliminating $\cot c$ by (1), $\cos B'=\cot^2 a/\cos B$. Eliminating $\cos B'$ by (4), $2\cos^2 B-1=-(\cot^2 a/\cos B)$. Putting $\cos B=y$ and substituting the numerical value for $\cot^2 a$, $y^3=\frac{1}{2}y+.094118=0$. Finding the two positive roots of this equation, $y=.5815$ and $.2056$. From the first root $B=54^\circ\ 27'=3$ hours, 37 minutes, 48 seconds. Length of day$=4B=14$ hours,

31 minutes, 12 seconds. From the second root, $B=78°\ 8'=5$ hours, 12 minutes, 32 seconds. Length of day=20 hours, 50 minutes, 8 seconds.

Values of B substituted in (1) give for the latitude 36° 44′ and 64° 39′.

III. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, O.

Let Z be the zenith, P the pole, N the north point of the horizon, S the place on the horizon where the sun rises, and S' the place of the sun on prime vertical when it has moved half the way to the meridian. Also let ϕ=the required latitude, δ=the sun's declination=23° 27′ 15″, and P=the hour angle of the sun when rising. Then without allowing for refraction and semi-diameter, we get from the spherical triangles PNS and PZS', $-\cos P=\tan\phi\tan\delta$......(1); and $\cos\frac{1}{2}P=\cot\phi\tan\delta$......(2). From (1) and (2) we have $\tan^3\phi-(1/\tan\delta)\tan^2\phi+2\tan\delta=0$, or $\tan^3\phi-2.31224\tan^2\phi-.18704=0$. Whence $\tan\phi=2.27614$, and $\phi=$ 66° 16′ 54″ the required latitude. From (2) $P=158°\ 1'\ 28''$. Hence from sunrise to noon is 10 hours, 32 minutes, 5.8 seconds, and the length of the day is 21 hours, 4 minutes, 11 seconds.

Solved by *J. SCHEFFER* with result, latitude 36° 43′ 31″, and length of day 14 hours, 31 minutes, 6 seconds.

61. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

The product of n numbers, each the sum of four squares, may be expressed as the sum of four squares in $(48)^{n-1}$ different ways.

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

In Vol. II, No. 2, page 47, it is demonstrated that the product of two numbers each the sum of four squares may be expressed as the sum of four squares in 48 different ways.

Let $a_1^2,\ a_2^2,\ a_3^2,\ a_4^2 \ldots\ldots a_n^2$ be the n squares.

Then $a_1^2a_2^2=(m_1^2+m_2^2+m_3^2+m_4^2)$ in 48 ways.

$$a_1^2a_2^2a_3^2=(m_1^2a_3^2+m_2^2a_3^2+m_3^2a_3^2+m_4^2a_3^2)=(n_1^2+n_2^2+n_3^2+n_4^2)$$
in $4\times48\times48=2^2.48^2$ ways.

$$a_1^2a_2^2a_3^2a_4^2=(n_1^2a_4^2+n_2^2a_4^2+n_3^2a_4^2+n_4^2a_4^2)=(o_1^2+o_2^2+o_3^2+o_4^2)$$
in $4\times2^2.48^2\times48=2^4.48^3$ ways.

$a_1^2a_2^2a_3^2a_4^2a_5^2=(p_1^2p_2^2p_3^2p_4^2)$ in $4\times2^4.48^3\times48=2^6.48^4$ ways.

$$\therefore\ a_1^2a_2^2a_3^2a_4^2\ldots\ldots a_n^2=(z_1^2+z_2^2+z_3^2+z_4^2)$$
in $2^{2n-4}.48^{n-1}$ ways $=(2)^{2(n-2)}.(48)^{n-1}$ ways.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

98. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A poor man borrowed \$20.00 which he repaid in eleven monthly installments of \$2.00 each; what was the annual rate of interest (reckoned as simple interest)?

99. If 300 cats catch 300 rats in 300 minutes, how many rats will 100 cats catch in 100 minutes? [From *Milne's Practical Arithmetic.*]

*** Solutions of these problems should be sent to B. F. Finkel, not later than Sept. 10.

ALGEBRA.

88. Proposed by E. S. LOOMIS, Ph. D., Professor of Mathematics in Cleveland West High School, Berea, O.

(1). The Indemnity Savings and Loan Company made two loans of \$1000 each to "A", one of its borrowers, under the following terms: In the first loan "A" agrees to cancel the \$1000 by making 120 payments of \$13.50, the first payment to be considered as made on the first of the month in which the loan is made, and the 119 subsequent payments to be made on the first of each subsequent month; in the second loan "A" agrees to cancel the \$1000 by making 120 payments of \$13.50, the first payment being made on the first of the month following the loan, and the 119 subsequent payments being made on the first of the subsequent months, Does the Company sustain any loss in earnings by the second loan over the first loan, and if so how much, and when is (or are) this loss (or these losses) sustained, the rate of interest in each loan being considered as $10\frac{1}{2}\%$ per annum?

(2). Deduce a formula for each case of proposition (1), by means of which one can find the balance of the loan uncancelled at the end of *any* month, *if* the loan is fully cancelled in 120 payments.

*** Solutions of this problem should be sent to J. M. Colaw, not later than Sept. 10.

GEOMETRY.

99. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy; Ohio State University, Athens, Ohio.

Find the locus of the vertices of all right cones which have the same given ellipse as a base.

100. Proposed by CHARLES C. CROSS, Libertytown, Md.

O, O_1, O_2, O_3 are the centers of the inscribed and three escribed circles of a triangle ABC. Prove $AO.AO_1.AO_2.AO_3=AB^2.AC^2$.

101. Proposed by E. W. MORRELL, A. M., Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

AB is the diameter of a circle and Q_0 any point on the circumference; Q_1, Q_2, Q_3.... are the points of bisection of the arcs AQ_0, AQ_1, AQ_2.... Prove that $BQ_1, BQ_2, BQ_3 \dots BQ_n=OA^n.(AQ_0/AQ_n)$.

*** Solutions of these problems should be sent to B. F. Finkel, not later than Sept. 10.

CALCULUS.

78. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Investigate value $\left(\frac{\tan x}{x}\right)^{1/x^n}$ where x is 0 and n has consecutive values 1, 2, 3, 4, ... Is there any law governing the different results? When $n=1$, result is 1; when $n=2$, result is $e^{\frac{1}{3}}$; $n=3$, gives α, etc.

79. Proposed by GEORGE LILLEY, Ph. D., Professor of Mathematics, University of Oregon, Eugene, Ore.

Find the area included between $y=\sin^{\pi} x+\cos^{\varepsilon} x$; $y=\pi e(\sin^{\pi} x\cos^{\varepsilon} x)$ and the length of its boundary, true to six decimal places, when $\pi=3.14159$, $e=2.7182$.

*** Solutions of these problems should be sent to J. M. Colaw, not later than Sept. 10.

MECHANICS.

71. Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.

Three men own a sphere of gold the density of which varies as the square of the distance from the center. If two segments be cut off each one inch from the center of the sphere it will be divided into three parts of equal value. Determine the diameter of the sphere.

72. Proposed by REV. A. L. GRIDLEY, Pastor of First Congregational Church, Kidder, Mo.

Prove that the motion of a ball falling through the earth influenced by gravity alone would be similar to the motion of a pendulum.

*** Solutions of these problems should be sent to B. F. Finkel, not later than Sept. 10.

DIOPHANTINE ANALYSIS.

71. Proposed by A. H. BELL, Hillsboro, Ill.

Find five numbers such that the product of any two plus 1 will equal a square.

72. Proposed by H. C. WILKES, Skull Run, W. Va.

Given $x^2+y^2+z^2=p^2+q^2+r^2$, to find unequal integral values for x, y, z, p, q, and r.

73. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find integral values for x and y in $\left(\begin{array}{l}2x^2-y^2=\square, \\ 2y^2-x^2=\square,\end{array}\right)$

*** Solutions of these problems should be sent to J. M. Colaw, not later than Sept. 10.

AVERAGE AND PROBABILITY.

67. Proposed by **HENRY HEATON, M. Sc.**, Atlantic, Ia.

A witness in court who undertook to recognize the signature of an individual failed four times in succession. What is the probability that he was correct the fifth time? An actual occurrence.

68. Proposed by **J. K. ELLWOOD, A. M.**, Principal of Colfax School, Pittsburg, Pa.

What are the odds against throwing 7 or 11 at one throw with two dice ?

*** Solutions of these problems should be sent to B. F. Finkel, not later than Sept. 10.

MISCELLANEOUS.

64. Proposed by **G. B. M. ZERR, A. M., Ph. D.**, President and Professor of Mathematics, The Russell College, Lebanon, Va.

Find the caustic by reflection of an hyperbola, the bright point being the center.

65. Proposed by **J. M. COLAW, A. M.**, Monterey, Va.

Three circles, radii in ratio 1, 3, 5, are tangent externally and enclose one acre ; what are the radii ?

*** Solutions of these problems should be sent to J. M. Colaw, not later than Sept. 10.

EDITORIALS.

Professor Fellows has been elected Professor of Mathematics in the University of Missouri.

Dr. George Bruce Halsted is spending the summer in Japan, his visit to that country being in the interest of mathematics.

Dr. Alexander Ziwet, of the University of Michigan, has been granted one years leave of absence. Dr. Ziwet expects to spend the time abroad.

Dr. G. A. Miller, of Cornell University, gave a course on Permutation Groups during the first term of the summer quarter, at the University of Chicago.

Prof. John B. Faught, of the Indiana University, has been assigned a $600 Fellowship at the University of Pennsylvania, where he expects to go the coming year to do graduate work.

Prof. T. U. Palmer, of the University of Alabama, and Professor Drope, of the University of Arkansas, are doing advanced work in mathematics at the University of Chicago, during the summer quarter.

Prof. G. B. M. Zerr has been elected Principal of the East Chester High School. Professor Zerr will now be located only a few miles from the University of Pennsylvania, to whose mathematical library he will have access, and of which he will make good use.

BOOKS AND PERIODICALS.

Elements of Trigonometry with Tables. By Herbert C. Whitaker, Ph. D. (University of Pennsylvania), Central Manual Training School, Philadelphia, Penn. 8vo Cloth, xvi + 182 pages. Price, $1.00. Philadelphia: D. Anson Partridge.

Among the many noteworthy features of this book are: The concise and accurate statement of definitions; a table of Circular Measure, i. e., where degrees and minutes are reduced to radians; clear presentation of fundamental conventions; a brief but comprehensive discussion of the Theory of Logarithms; and an excellent introduction to the study of Complex Quantities and Hyperbolic Functions. The discussion of the Theory of Logarithms and the setting forth of the method by which Napier invented his system will be of great value to all students of Trigonometry and Algebra. The work, in every way, is worthy of the highest patronage by teachers who are contemplating a change of text-books on this subject. B. F. F.

Prismoidal Formulæ and Earthwork. By Thomas U. Taylor, C. E. (University of Virginia), M. C. E. (Cornell University), Associate Member of the American Society of Civil Engineers; Member of the American Mathematical Society; Professor of Applied Mathematics, University of Texas. First edition, first thousand. 8vo Cloth, x + 102 pages. Price, $1.50. New York: John Wiley & Sons.

The object of this admirable treatise on *Prismoidal Formulæ and Earthwork* is to present a ready method of estimating the usual quantities in earthwork computations by graphical methods. The method outlined by the author applies to the majority of earthwork calculations, whether the volume be calculated by the Newtonian or average end-area formula. In this respect the work will be of great value to the civil engineer. The author, in writing the book, has also discovered the original authors of the usual formalæ associated with the prismatoid and ascribes the honor to Newton, Hirsch, Koppe and Kinklin as can be verified by history. The introduction to the work is of great historic interest and value. B, F. F.

Lectures on the Geometry of Position. By Theodore Reye, Professor of Mathematics in the University of Strassburg. Translated and Edited by Thom- F. Holgate, M. A., Ph. D., Professor of Applied Mathematics in Northwestern University. Part I. 8vo Cloth, xix + 248 pages. Price, $2.25, *net.* New York: The Macmillan Co.

It is believed that Professor Reye's work, *Geometrie der Lage*, is the best in any language. In translating the first part of this incomparable work into English, the translator has placed within easy reach of the English-speaking student the most refined discus-

sion of pure geometry extant. There is scarcely another field in mathematics at once so interesting and so fruitful of discovery and yet lacking much of the abstruseness characterizing other modern mathematical subjects as is Projective Geometry, or Geometry of Position. Notwithstanding the fact that the subject is the most stimulating branch of mathematics and admits of various beautiful and convenient applications to technical and natural sciences, yet it has not thus far received the attention of teachers of mathematics it so richly deserves. It is to be hoped that this very excellent translation by Dr, Holgate will be the means of introducing the subject in the course of mathematics in every college in this country. The translation, in general, is rather liberal than literal, thus presenting in good, readable English, without destroying the charm of the original writing, the geometric ideas contained in the text.

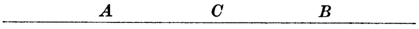
Some very decided improvements over the original have been made, viz., The articles have been numbered; the examples set at the end of the lectures to which they are related and a few new ones added; explanatory notes have been inserted where they seemed necessary or helpful, and an index has been compiled.

The following is the table of contents:

Lecture I—The Methods of Projection and Section—The Six Primitive Forms of Modern Geometry: Lecture II—Infinitely Distant Elements—Correlation of the Primitive Forms to one another. In this lecture are set down the conventions, interpretations or assumptions respecting the "point infinity," "the line infinity." As there has been, in the past, quite a good deal said in the *Monthly* about the fallacious argument introduced in modern mathematics it will be well for those who wish to gain thorough knowledge of the modern notions of "the point infinity" and the line infinity to read carefully this second lecture of Reye.

From the following, one of a number of considerations, we assume that every straight line has one point and only one point at infinity:

Let AB be a straight line. Take C anywhere between A and B. Also consider the



directions AB positive and BC negative. Then AC/CB is positive for all positions between A and B. When C is at A, AC/CB is 0; when C is at B, AC/CB is ∞. When $AC=CB$, $AC/CB=1$, and AB is bisected internally at C. When C is at the right of B, AC/CB is negative, and when C is at an infinite distance from B, AC/CB is -1. When C is to the left of A, AC/CB is negative, and when C is at an infinite distance from A, AC/CB is -1. Since in going in both directions, an infinite distance from A, we get the same quotient, -1, we conclude there is just one such point of bisection [corresponding to the internal point of bisection C when C is to the right of A and to the left of B], or the point of infinity. This forces us to assume that the straight line is a continuous or closed line, its extremities meeting in the point infinity.

The ancients had no use for the above notions and so did not insert them in their mathematics. Whether or not the physical properties of a straight line agree with the above notions can neither be proved or disproved. The ancients said, two straight lines are parallel when they have no point in common, however far they be produced. The moderns say: Two lines are parallel when they have only one point in common, viz., the point at infinity, The modern conception of parallelism has distinct advantages over the ancient one, in that "first, many theorems can be enunciated in a perfectly general way for which otherwise exceptions would always have to be cited, and second, many apparently difficult theorems can, in accordance with this view, be comprised in a single statement."

All the infinite points of a plane lie on a straight line, viz., the line at infinity; since it is intersected by every line of the plane in only one point, the point infinity. A curved line may have, in common with a straight line, more than one point.

Two parallel planes have only one line in common, viz., the line infinity.

All infinity points and lines in space lie in a plane, viz., the plane at infinity; since it is intersected by every straight line in only one point and by every plane in a straight line.

To see to what great advantage these principles are turned, one needs to read carefully the whole of Reye's great work.

Lecture III.—The Principles of Reciprocity or Duality—Simple and Complete n-Points, n-Sides, n-Edges, etc.

Lecture IV.—Correlation of Complete n-Points, n-Sides, and n-Edges to one another—Harmonic Forms—Examples; Lecture V.—Projective Properties of One-Dimensional Primitive Forms—Examples; Lecture VI.—Curves, Sheaves and Cones of the Second Order—Examples; Lecture VII.—Deductions from Pascal's and Brianchon's Theorems—Examples; Lecture VIII.—Pole and Polar with respect to Curves of the Second Order; Lecture IX.—Diameters and Axes of Curves of the Second Order—Algebraic Equations of these Curves—Examples; Lecture X.—Regular and Ruled Surfaces of the Second Order—Examples; Lecture XI.—Projective Properties of Elementary Forms—Examples; Lecture XII.—Involution—Examples; Lecture XIII.—Metric Relations of Involution—Foci of Curves of the Second Order—Examples; Lecture XIV.—Problems of the Second Order Imaginary Elements—Examples; Lecture XV.—Principal Axes and Planes of Symmetry—Focal Axes and Cyclic Planes of a Cone of the Second Order—Examples; Appendix.—Principal of Reciprocal Radii—Ruled Surfaces of the Third Order—Quadrangles and Quadrilaterals which are Self-Polar with respect to Conic Sections—Nets and Webs of Conic Sections; Index. B. F. F.

Collier's Weekly. An Illustrated Journal of Art, Literature, and Current Events. Special price to close of year, $1.00. 521-549 West Thirteenth Street, New York City.

In view of the widespread interest in the present war, *Collier'r Weekly* is especially popular. Its correspondents with the army and navy furnish most interesting letters each week, while the war pictures, in the illustrated bulletin of the week's news from the front, are unsurpassed. J. M. C,

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. AUGUST-SEPTEMBER, 1898. No. 8-9.

BIOGRAPHY.

RENÉ DESCARTES.

BY B. F. FINKEL.

RENÉ DESCARTES, the first of the modern school of mathematicians, was born at La Haye, a small town on the right bank of the Creuse and about midway between Tours and Poitiers, on March 31st, 1596, and died at Stockholm, on February 11th, 1650. "The house is still shown where he was born, and a *métairie* about three miles off still retains the name of Les Cartes. His family on both sides was of Poitevin descent and had its headquarters in the neighboring town of Châtterault, where his grandfather had been a physician. His father, Joachim Descartes, purchased a commission as counsellor in the Parlement Rennes and thus introduced the family into that demi-noblesse of the robe of which, in stately isolation between the bourgeoisie and the high nobility, maintained a lofty rank in the hierarchy of France. For one-half of each year required for residence the elder Descartes removed, with his wife, Jeanne Brochard, to Rennes. Three children, all of whom first saw the light at La Haye, sprang from the union,—a son, who afterwards succeeded to his father in the Parlement, a daughter who married a M. du Crevis, and a second son, René. His mother, who had been ailing beforehand, never recovered from her third confinement; and the motherless infant was intrusted to a nurse, whose care Descartes in after years remembered by a small pension."*

***Britannica Encyclopedia*, Ninth Edition.

RENE DESCARTES.

By courtesy of The Open Court.

Descartes, who early showed an inquisitive mind, was called by his father, "my philosopher." At the age of eight, Descartes was sent to the school of La Flèche, which Henry IV had lately founded and endowed for the Jesuits, and here he continued from 1604 to 1612. Of the education here given, of the equality maintained among the pupils, and of their free intercourse, he spoke at a later period in terms of high praise. Descartes himself enjoyed exceptional privileges. His feeble health excused him from the morning duties, and thus early he acquired the habit of matutinal reflection in bed, which clung to him throughout life. When he visited Pascal in 1647, he told him that the only way to do good work in mathematics and to preserve his health was never to allow any one to make him get up in the morning before he felt inclined to do so. Even at this period he had begun to distrust the authority of tradition and his teachers.

Two years before leaving school (1610) he was selected as one of twenty-four gentlemen who went forth to receive the heart of the murdered king as it was borne to its resting place at La Flèche. During the winter of 1612, he completed his preparations for the world by lessons in horsemanship and fencing; and then in the spring of 1613 he started for Paris to be introduced to the world of fashion. Fortunately the spirit of dissipation did not carry him very far, the worst being a passion for gaming. Here through the medium of the Jesuits he made the acquaintance of Mydorge, one of the foremost mathematicians of France, and renewed his schoolboy friendship with Father Mersenne, and together with them he devoted the two years of 1615 and 1616 to the study of mathematics.

"The withdrawal of Mersenne in 1614 to a post in the provinces was the signal for Descartes to abandon social life and shut himself up for nearly two years in a secluded house of the Faubourg St. Germain. Accident, however, betrayed the secret of his retirement; he was compelled to leave his mathematical investigations and to take a part in entertainments, where the only thing that chimed in with his theorizing reveries was the music. The scenes of horror and intrigue which marked the struggle for supremacy between the various leaders who aimed at guiding the politics of France made France no fit place for a student and held out little honorable prospect for a soldier. Accordingly, in May, 1617, Descartes, now twenty-one years of age, set out for the Netherlands, and took service in the army of Prince Maurice of Orange, one of the greatest generals of the age, who had been engaged for some time in a war with the Spanish forces in Belgium. At Breda, he enlisted as a volunteer, and the first and only pay which he accepted he kept as a curiosity through life. There was a lull in the war; and the Netherlands were distracted by the quarrels of Gomarists and Arminians. During the leisure thus arising, Descartes one day, as he roved through Breda, had his attention drawn to a placard in the Dutch tongue; and as the language of which he never became perfectly master, was then strange to him, he asked a bystander to interpret it in either French or Latin. The stranger, who happened to be Isaac Beeckman, principal of the College of Dort, offered with some surprise to do so into Latin, if the inquirer would bring him a so-

lution of the problem,—for the advertisement was one of those challenges which the mathematicians of the age, in the spirit of the tournament of chivalry, were accustomed to throw down tc all comers, daring them to discover a geometrical mystery known as they fancied to themselves alone. Descartes promised and fulfilled; and a friendship grew up between him and Beeckman—broken only by the literary dishonesty of the latter, who in later years took credit for the novelty contained in a small essay on music (Compendium Musicae) which Descartes wrote at this period and intrusted to Beeckman."*

The unexpected test of his mathematical attainments afforded by the solution of the problem referred to, its solution costing him only a few hours study, made the uncongenial army life distasteful to him, but under family influence and tradition, he remained a soldier, and was pursuaded at the commencement of the thirty years' war to volunteer under Count de Bucquoy in the army of Bavaria. The winter of 1619, spent in quarters at Neuburg on the Danube, was the critical period in his life. Here, in his warm room (*dans un poele*), he indulged those meditations which afterwards led to the *Discours de la Méthode* (Discourse of Method). It was here that, on the eve of St. Martin's day, November 10, 1619, he "was filled with enthusiasm, and discovered the foundations of a marvelous science."

He retired to rest with anxious thoughts of his future career, which haunted him through the night in three dreams, that left deep impressions on his mind. "Next day," he says, "I began to understand the first principles of my marvelous discovery." Thus the date of his philosophical conversion is fixed to a day. This day marks the birth of modern mathematics. His discovery, viz., the coöperation of ancient geometry and algebra, is epoch-making in the history of mathematics.

It is frequently stated that Descartes was the first to apply algebra to geometry. This statement is not true, for Vieta and others had done this before him, and even the Arabs sometimes used algebra in connection with geometry. "The new step that Descartes did take was the introduction into geometry of an analytical method based on the notion of variables and consonants, which enabled him to represent curves by algebraic equations. In the Greek geometry, the idea of motion was wanting, but with Descartes it became a very fruitful conception. By him a point was determined in position by its distances from two fixed lines or axes. These distances varied with every change of position in the point. This geometric idea of *co-ordinate representation* together with the algebraic idea of *two variables in one equation* having an indefinite number of simultaneous values, furnished a method for the study of loci, which is admirable for the generality of its solutions. Thus the entire conic sections of Appollonius is wrapped up and contained in a single equation of the second degree."†

"Descartes found in mathematics, as did Kant and Comte, the type of all faultless thought; and he proved his appreciation of his insight by the invention

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**Encyclopedia Britannica*, Ninth Edition.
†*Cajori's History of Mathematics*.

of a new symbolic mechanism and artifice for the applications of algebra to geometry (*Analytic Geometry*, as it is now called, which, in a growing sense, let it be said, existed before him), and by his discoveries in the theory of equations, which were fundamental in their importance."*

After a short sojourn in Paris, Descartes moved to Holland, then at the height of its power. There for twenty years he lived, giving up all his time to philosophy and mathematics. Science, he says, may be compared to a tree; metaphysics is the root, physics is the trunk, and the three chief branches are mechanics, medicine, and morals, these forming the three applications of our knowledge, namely, to the external world, to the human body, and to the conduct of life; and with these subjects alone his writings are concerned.

He spent the time from 1629 to 1633 writing *Le Monde*, a work embodying an attempt to give a physical theory of the universe; but finding its publication likely to bring on him the hostility of the Church, and having no desire to pose as a martyr, he abandoned it. The incomplete manuscript was published in 1664.

He then devoted himself to composing a treatise on universal science; this was published at Leyden in 1637 under the title Discourse de la méthode pour bien conduire sa raison et chercher la verité dans les sciences, and was accompanied with three appendices entitled La Dioptrique, Les Méléores, and La Géométrie. It is from the last of these that the invention of analytical geometry dates. In 1641, he published a work called Meditations, in which he explained at some length his views of philosophy as sketched out in the *Discourse*. In 1644, he issued the Principia Philosophiae, the greater part of which was devoted to physical science especially the laws of motion and the theory of vortices. In his theory of vortices, he commences with a discussion of motion; and then lays down ten laws of nature, of which the first two are almost identical with the first two as laid down by Newton. The remaining eight are inaccurate. He next proceeds to a discussion of the nature of matter which he regards uniform in kind though there are three forms of it. He assumes that the matter of the universe is in motion, that this motion is constant in amount, and that the motion results in a number of vortices. He states that the sun is the cênter of an immense whirlpool of this matter, in which the planets float and are swept round like straws in a whirlpool of water.

Each planet is supposed to be the center of a secondary whirlpool by which its satellites are carried, and so on. All of these assumptions are arbitrary and unsupported by any investigation. It is a little strange that a man who began his philosophical reasonings by doubting all things and finally coming to *cogito, ergo sum* should have made assumptions so groundless.

While Descartes was a philosopher of a very high type, yet his fame will ever rest on his researches in mathematics. The first important problem solved by Decartes in his geometry is the problem of Pappus, viz.: "Given several straight lines in a plane, to find the locus of a point such that perpendiculars, or,

*The Open Court, August. 1898.

more generally, straight lines at given angles, drawn from the point to the given lines, shall satisfy that the product of certain of them shall be in given ratio to the product of the rest." "The most important case of this problem is to find the locus of a point such that the product of the perpendiculars on m given lines be in a constant ratio to the product of the perpendiculars on n other given straight lines. The ancients had solved this geometrically for the case $m=1$, $n=1$, and the case $m=1$, $n=2$. Pappus had further stated that if $m=n=2$, the locus was a conic, but he gave no proof; Descartes also failed to prove this by pure geometry, but he showed that the curve was represented by an equation of the second degree, that is, was a conic; subsequently Newton gave an elegant solution of the problem by pure geometry."*

In algebra, Descartes expounded and illustrated the general methods of solving equations up to those of the fourth degree (and believed that his method could go beyond), stated the law which connects the positive and negative roots of an equation with the change of signs in the consecutive terms, known as Descartes' Law of Signs, and introduced the method of indeterminate coefficients for the solution of equations.

In appearance, Descartes was a small man with large head, projecting brow, prominent nose, and black hair coming down to his eyebrows. His voice was feeble. Considering the range of his studies he was by no means widely read, had no use for Greek, as is shown by his disgust when he found that Queen Christina devoted some time each day to its study, and despised both learning and art unless something tangible could be extracted therefrom. In philosophy, he did not read much of the writings of others. In disposition, he was cold and selfish, He never married, and left no descendants, though he had one illegitimate daughter, Francine, who died in 1640, at the age of five.

In 1649, through the instigation of his close personal friend, Chanut, he received an invitation to the Swedish court, and in September of that year he left Egmond for the north. Here, on the 11th of February, 1650, he died of inflamation of the lungs brought about by too close devotion to the sick-room of his friend Chanut, who was dangerously ill with the same disease.

**Ball's Short Account of the History of Mathematics.*

ON SEVERAL POINTS IN THE THEORY OF THE GROUPS OF A FINITE ORDER.*

By DR. G. A. MILLER.

1. *Definition.* The most prominent condition which the operators of a group must satisfy is that the product of any two of them is equivalent to a single operator contained in the group. When the operators of the group are represented by substitutions it is not necessary to add any additional condition, *i. e.* a set of substitutions which includes every substitution that is obtained by multiplying any two of the set together or by squaring any one of them forms a group, provided that no two of the substitutions are identical.

It is however not customary to call *any* set of operators that satisfies the given condition a group : *e. g.* the three numbers 0, 1, -1 clearly satisfy this condition while they would not generally be said to form a group when they are combined by multiplication. Some writers seem to make the definition of group so general as to include this case,† but the theory of the groups of operators which satisfy this general definition without also satisfying the more restricted ones remains to be developed.

In addition to the given conditions the operators of a group are usually required to satisfy the associative law, and the laws that the product of any two of them is completely determined by the operators and the method of combining them and that from the equations $ab=ac$, $\alpha\beta=\gamma\beta$ it must follow respectively that $b=c$, $\alpha=\gamma$; where a, b, c, α, β, γ represent any operators of the group.

While the method of combining the operators of a group is generally called multiplication yet it should not be inferred that any restrictions are imposed upon this method except those given above, so that the term multiplication in this connection merely implies some definite law of combination : *e. g.* if we combine the following n numbers, 0, 1, 2, 3, ... , $n-1$, by adding any one to each of them, their positive remainders with respect to modulus n will determine the cyclical group of order n and every cyclical group can be represented in this way. If we combine these numbers by multiplication, taking their positive remainders according to the same modulus, they do *not* form a group in the usual sense of this term. It is thus clear that the group property is not inherent in a set of operators but, that it exists, in part, in the law of combination.

As may be inferred from the heading, these remarks relate to the groups of a finite order. This restriction seemed appropriate for this occasion since my investigations have been confined to this class of groups. In recent years the groups of an infinite order have received a great deal of attention. Among the investigators of the discontinuous groups of an infinite order Poincaré, Klein and Fricke are the most prominent while Sophus Lie is preëminent among the many investigators of continuous groups. After this glance at these two sturdy

*Read before the *Oliver Mathematical Club* of Cornell University as initiatory paper, February, 1898.

†Cf. Klein's Ikosaeder, page 5; Miss Scott's Modern Analytical Geometry, page 261; etc.

offsprings of our subject we proceed to give a short sketch of its development.

Historical Sketch. While the group concept appears early in the development of mathematics yet it was not until substitution groups began to be studied that this concept began to receive considerable attention and that its fundamental importance began to be recognized. Writers do not agree in regard to the founder of the theory of substitutions. Some ascribe this honor to Lagrange, others to Abel, others to Galois, and still others to Cauchy. In recent years a considerable number, under the leadership of Burkhardt,* have been contending that the origin of the theory of substitutions is found in an Italian work bearing the title "*Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto*, di Paolo Ruffini," Bologna, 1799.

This work contains some fundamental concepts such as transitivity, intransitivity, primitivity, etc., that are generally attributed to the later works of Cauchy. Its proof that the general equation whose degree exceeds four is not algebraically solvable lacks rigor. The first vigorous proof of this important theorem, which engaged the attention of mathematicians for centuries, was given by Abel and it was published in the first volume of Crelle, the oldest German mathematical paper still living. As this proof was partly based upon the theory of substitutions it drew considerable attention to this subject. This seems to be the main reason why some writers, Hagen for example, call Abel one of the founders of the theory of substitutions.†

The next great impulse to the study of this theory was given by that very remarkable young mathematician Galois, who proved that every algebraic equation belongs to a group and that its solvability by the extraction of roots depends upon the factors of composition of this group. This discovery united the theory of equations very closely with the theory of substitution groups. The French mathematician Jordan has been especially prominent in pointing out results which depend upon this relation. The subject is one of great difficulty and it will probably furnish a fertile field of investigation for many years to come.

The claims that Cauchy was the founder of the theory of substitutions seem to be based upon the fact that he was the first to write extensively on this subject apart from any direct applications. His *Exercices d' Analyse*, vol. 3, (1844), contains the first systematic treatment of this theory and contributed very largely towards making it known to a larger class of students. Although some of the concepts which have generally been attributed to him have recently been traced to earlier writings yet there remains much which is undoubtedly due to Cauchy and some of these facts are of fundamental importance.

In recent years there has been developing an offspring from substitution groups, which resembles the parent more closely than the two mentioned above. I refer to the operation or abstract groups. In the first volume of the *American Journal of Mathematics* Cayley calls attention to the fact that a group is an ab-

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**Schlomilch's Zeitschrift*, 1892, supplement, page 159.

†*Synopsis der Mathematik*, page 281.

stract concept and that it should be defined "by means of the laws of combination of its symbols." A few years later Dyck published several articles in the *Mathematische Annalen* in which he laid the foundation of a part of the theory of abstract groups.

As many of the properties of these groups can be readily studied by means of substitution groups their development has given a new impulse to the study of substitution groups. This is the modern impulse and it is to be hoped that it will lead to great activity in this line. The bulk of the recent publications on groups of a finite order have been along the lines of abstract groups and these groups promise to remain a fertile field of investigation for a number of years at least.

Standing Problems. While knowledge is cosmopolitan and scientific discoveries belong to the world rather than to a particular nation yet it does not behoove a great nation or even a great university to be satisfied to borrow all its scientific facts from the rest of the world. Such a state of affairs indicates stagnation and is evidence either of inability or of want of patriotism. It also deprives the students of the inspiration and joy which attend the discovery of important scientific facts that tend to contribute something towards enriching all future generations.

It is evident that the standing problems of a comparatively new subject can be reached more readily and are generally less difficult than those of older subjects. Hence the newer subjects are the more inviting to the young investigator. While our modern mathematical journals do a great deal towards aiding the student to find desirable unsolved problems yet some of the simpler ones are solved at such a rapid rate that it is scarcely possible to keep in touch with a large portion of the region in which these discoveries are being made.

It has been observed for a long time that the problem to find all the substitution groups of any degree would be of the greatest importance to algebra.* This problem is far from a complete solution. It has been solved for all the degrees less than eleven but the methods employed are not suitable for very large degrees and they throw very little light on the general problem. The general theorems on this subject are still few, and the progress that has been made in recent years under the inspiration of the great prize of the Paris Institute gives evidence of the great difficulty of the problem rather than any hope of its early solution.

In recent years all the simple groups of the finite continuous groups have been found. It would be a great step forward if all the simple groups of the groups of a finite order could also be found. Several new infinite systems of such groups have recently been discovered by Professors Moore and Burnside, and by Dr. Dickson but the progress towards the complete solution of this problem is exceedingly slow. Any discoveries that throw light on this subject are of great interest.

As problems of somewhat smaller interest in themselves we may mention

*Cf. Serret's *algebra superieure*, 3rd edition, page 256.

the proof of the existence or non-existence of simple groups of an odd order or of order $p^{\alpha} q^{\beta}$, p and q being prime numbers; the superior limit of transitivity of primitive groups that do not contain the alternating group; the simplification of the methods of proving the solvability or the insolvability of a group, etc.

REPLY TO PROFESSOR FISK'S CRITICISM OF A CERTAIN FEATURE OF NICHOLSON'S CALCULUS.

By J. W. NICHOLSON, A. M., LL. D., Professor of Mathematics, Louisiana State University, Baton Rouge, La.

In the March number of the *Bulletin* is a brief review of my Calculus, by Professor Fiske, of which the following is an extract:

"In another note (A_3) at the end of the work the author critizes the grounds assigned by Byerly and by Rice and Johnston for making $d(dx)=0$. He contends that the differential of dx is zero, because dx as a variable is independent of x. This, of course, is not sound. If a variable y is independent of another variable x, it is true that we may still write

$$dy=\frac{dy}{dx}\,dx;$$

but the coefficient of dx is not a partial derivative, and dy, therefore, instead of being zero is indeterminate. In order that $d(dx)$ may be zero, we must assume that dx takes the same value for all values of x. This assumption, however, does not prevent our varying dx from one instant to another in a perfectly arbitrary manner."

As the question involved is an interesting and important one, and believing that Professor Fiske had not fairly presented my discussion of the point at issue, I wrote a brief reply to the above criticism, and sent it to the *Bulletin* for publication. Several weeks thereafter my reply was returned to me without publication and with the following additional stricture:

"The author fails to realize that if $dy=0$ when x goes from x to $x+dx$ then y is not completely independent of x, but has such a dependence that it does not alter when x alters."

The question involved is not whether dx is a quantity whose total differential is 0, but whether it is a quantity whose differential *with respect to x* is 0.

Of course if x and y are two variables which are independent of each other, and dx be the total differential of the one and dy that of the other, and $\frac{dy}{dx}$ is understood to mean the ratio of these differentials, $\frac{dy}{dx}$ is not 0 but indeterminate, as Professor Fiske says. Or again, under the same hypothesis, "if $dy=0$

when x goes from x to $x+dx$'' y would depend on x in the manner indicated by the last criticism.

But the point in question comes up in proving that $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, *i. e.* the differential coefficient of $\frac{dy}{dx}$ with respect to x, $=\frac{d^2y}{dx^2}$; and in this demonstration we have no occasion to consider whether dx is a quantity whose total differential is 0 or not. It is a differentiation with respect to x which is indicated by $\frac{d}{dx}$; and it is to be demonstrated that if this operation be performed on $\frac{dy}{dx}$, on the hypothesis that this symbol may be treated as a fraction whose terms are dy and dx, and the understanding that d^2y *represents the differential of* dy *with respect to* x, the result is $\frac{d^2y}{dx^2}$. Thus:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dx\,\frac{d}{dx}(dy)-dy\,\frac{d}{dx}(dx)}{dx^2}=\frac{d^2y}{dx^2},$$

because $d^2y=\frac{d}{dx}(dy).dx$, by definition, and $\frac{d}{dx}(dx)=0$, *for the same reason that the differential coefficient with respect to* x *of any variable which is independent of* x *is* 0. $\frac{d}{dx}(dx)$ does not mean the ratio of any variation that dx may be supposed to undergo, to dx, but the ratio of the variation which dx undergoes *in consequence of a variation in* x, to dx. To say that $\frac{d}{dx}(dx)=0$ is not therefore to say that dx is a constant, but merely that it undergoes no variation in consequence of a variation in x. Indeed, dx may have any value of x, and is therefore a variable independent of x, and being *independent*, it may be regarded and treated as an *absolute constant* except in cases where the independence of x would thereby be destroyed, as shown in my Calculus.

Precisely the same considerations are involved in the derivation of the equation $d^2y=f''(x)dx^2$ from $dy=f'(x)dx$.

The reply to Professor Fiske is therefore that in his equation, $dy=\frac{dy}{dx}dx$, for the case before us, viz: $d^2x=\frac{d}{dx}(dx).dx$, the coefficient of dx *is a partial derivative*.

The reply to the last criticism of the *Bulletin* is that in the case before us the dy is not any variation that y may be supposed to undergo while x varies from x to $x+dx$, but the variation which y undergoes in consequence of this variation in x, and this of course must be 0 if y is independent of x, whatever may be the value of y as x goes from x to $x+dx$.

The criticism in the *Bulletin* is therefore based upon a misconception of

the author's meaning, and is due to an apparent failure on the part of Professor Fiske to realize that the question is not what must be in order that $d(dx)$ may be 0, but what is in order that $\frac{d}{dx}(dx)$ may be 0.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

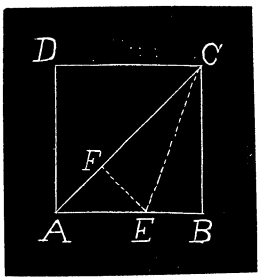
ARITHMETIC.

96. Proposed by RAYMOND SMITH, Tiffin, Ohio.

How many acres in a square field whose diagonal is 10 rods longer than the side ?

I. Solution by J. F. TRAVIS, Student at Ohio State University, Columbus, O.; EDWARD R. ROBBINS, Master of Mathematics, Lawrenceville School, Lawrenceville, N. J.; J. SCHEFFER, A. M., Hagerstown, Md.; F. R. HONEY, Ph. B., New Haven, Conn.; M. E. GRABER, Mt. Eaton; O.; WALTER HUGH DRANE, Professor of Mathematics, Jefferson College, Washington, Miss.; and JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Let $ABCD$ be the square field, and AC its diagonal. On AC lay off CF equal to BC. At F erect FE perpendicular to AC and intersecting AB in E. Draw EC. Then in the right triangles CFE and CBE, CB equals CF, by construction and CE is common. Hence, FE equals EB. In the right triangle AFE, the angle FAE is equal to 45°. Hence, the angle FEA equals 45°. Hence the side AF equals the side FE. Then



$$AB=(AE+EB)=[\sqrt{(AF^2+FE^2)}+EB]$$

$$=[\sqrt{(2EF^2)}+EB]=(EF\sqrt{2}+EB)=(\sqrt{2}+1)EB.$$

But $EB=AF=10$ chains. $\therefore$ $AB=10(\sqrt{2}+1)$, and area of the field $=AB^2=100(\sqrt{2}+1)^2=100(3+2\sqrt{2})=582.8427$ square rods, or 3.642 acres.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics, Chester High School, Chester, Pa.; and P. S. BERG, Superintendent of Schools, Larimore, N. D.

We will solve generally by making a=the excess of the diagonal over the side.

Let x=side of square field. Then $2x^2=(x+a)^2$.

Solving this equation for x, gives, $x=a(1\pm\sqrt{2})$.

$\therefore$ area$=x^2=a^2(3\pm2\sqrt{2})$.

Now substituting 10 for a, and we obtain

Area$=100(3\pm2\sqrt{2})=582.8427+$ square rods, or 17.157+ square rods,

$=3.6429+$ acres, or .10723+ acres.

[QUERY. How interpret the second result? M. A. G.]

[NOTE.—The answer to this query is simply this: The second result is not geometrically interpretable, for the reason that the negative value of x from which it is derived, is not geometrically intrepretable. The negative value of x satisfies the algebraic condition expressed by the equation and that alone. ED. F.]

III. Solution by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.; COOPER D. SCHMIDT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and CHARLES C. CROSS, Libertytown, Md.

The diagonal of a square=the side$\times\sqrt{2}$.

$\therefore$ side$\times\sqrt{2}$=side+10 rods, or side$(\sqrt{2}-1)=10$ rods, and side$=10\div(\sqrt{2}-1)$.

Hence $[10/(\sqrt{2}-1)]^2\div160=3.64+$, the number of acres.

97. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

In what time will \$4000 amount to \$5134.96, interest at 6% payable annually?

Solution by WILLIAM W. CHAMPLAIN, Wickford, R. I.

The time is evidently between 4 and 5 years.

The interest on \$4000 for 4 years, at 6% is	\$960.00
The interest on \$240 for (3+2+1) years=6 years is	86.40
	\$1046.40

\$1134.96−\$1046.40=\$88.56, the interest on \$4000 for the number of months and days exceeding 4 years, plus the interest on four unpaid installments of annual interest for the same period; that is, on \$4000+\$960 or \$4960. Interest on \$4960 at 6% for one year is \$297.60; to gain \$88.56 it would require $\frac{8856}{29760}$ of a year, or 3 months, 17.13 days.

$\therefore$ \$4000 would amount to \$5134.96, at 6% annual interest in 4 years, 3 months, 17.13 days.

Also solved by *G. B. M. ZERR*, and *P. S. BERG*.

98. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A poor man borrowed \$20 which he repaid in eleven monthly installments of \$2 each; what was the annual rate of interest (reckoned as simple interest)?

Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

The last installment would be paid in eleven months, which, therefore, would be the time the \$20 would be on interest, and the interest would be $\frac{11}{12}\times20\times$rate. The first installment would be on interest 10 months, and the interest on the installments would be $\frac{55}{12}\times2\times$rate. Then $\frac{220}{12}$ times the rate less $\frac{110}{12}$ times the rate would be \$2, or \$2 is $\frac{110}{12}$ times the rate. Hence the rate is $21\frac{9}{11}$

Also solved by *WALTER H. DRANE*.

[NOTE. P. S. Berg solved problem 95, but his solution reached us too late for credit in last issue. Problem 99, should read, "How many cats will catch 100 rats in 100 minutes?" Ed. F.]

ALGEBRA.

84. Proposed by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.

On the present electoral basis, if all the electoral votes of each state are cast solid for one or the other of two presidential candidates, how many combinations of states are possible for a total of 273 votes for the winning candidate?

No solution of this problem has been received.

85. Proposed by J. M. COLAW, A. M., Monterey, Va.

Sum the infinite series,

$$\frac{1}{1^2.2^2.3^2.4^2.5^2}+\frac{1}{2^2.3^2.4^2.5^2.6^2}+\frac{1}{3^2.4^2.5^2.6^2.7^2}+\text{, etc.}$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science in Chester High School, Chester, Pa.

Let $V_m=1/[n^2(n+1)^2\ldots\ldots(n+m)^2]$.

$$\text{Then } V_m=\frac{2m-1}{m^3}\left[\frac{1}{n^2.(n+1)^2\ldots.(n+m-1)^2}+\frac{1}{(n+1)(n+2)^2\ldots.(n+m)^2}\right]$$

$$+\frac{1}{m^3(m-1)}\left[\frac{2}{(n+1)^2\ldots.(n+m-1)^2}-\frac{1}{n^2\ldots.(n+m-2)^2}\right.$$

$$\left.-\frac{1}{(n+2)^2\ldots.(n+m)^2}\right]$$

Now let $S_m=\Sigma V_m$.

$$\therefore\ S_m=\frac{2m-1}{m^3}\left[S_{m-1}+\left(S_{m-1}-\frac{1}{m^2\,!}\right)\right]$$

$$+\frac{1}{m^3(m-1)}\left[\left(2S_{m-2}-\frac{2}{(m-1)^2\,!}\right)-S_{m-2}-\left(S_{m-2}-\frac{1}{(m-1)^2\,!}-\frac{1}{m^2\,!}\right)\right].$$

$$\therefore\ S_m=\frac{4m-2}{m^3}.S_{m-1}-\frac{2m-1}{m^3.(m^2)!}-\frac{1}{m^3(m-1)}\left[\frac{1}{(m-1)^2!}-\frac{1}{m^2\,!}\right]$$

$$=\frac{4m-2}{m^3}S_{m-1}-\frac{2m-1}{m^3.(m^2)\,!}-\frac{m^2-1}{m^3(m-1).(m^2)\,!}$$

$$=\frac{4m-2}{m^3}S_{m-1}-\frac{1}{m^3.(m^2)!}\left[(2m-1)+(m+1)\right]$$

$$=\frac{4m-2}{m^3}S_{m-1}-\frac{1}{m^2.(m^2)!}=\frac{1}{m^2}\left[\frac{4m-2}{m}S_{m-1}-\frac{3}{m^2\,!}\right].$$

Now $S_0=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.\frac{\pi^2}{6}$.

$$\therefore\ S_1=\frac{1}{1^2.2^2}+\frac{1}{2^2.3^2}+\frac{1}{3^2.4^2}+\ldots.\frac{\pi^2}{3}-3.$$

$$S_2=\frac{1}{1^2.2^2.3^2}+\frac{1}{2^2.3^2.4^2}+\frac{1}{3^2.4^2.5^2}+\ldots.\frac{\pi^2}{4}-\tfrac{39}{16}.$$

$$S_3=\frac{1}{1^2.2^2.3^2.4^2}+\frac{1}{2^2.3^2.4^2.5^2}+\ \ldots\frac{5\pi^2}{54}-\tfrac{197}{216}.$$

$$S_4=\frac{1}{1^2.2^2.3^2.4^2.5^2}+\frac{1}{2^2.3^2.4^2.5^2.6^2}+\ldots.\frac{35\pi^2}{1728}-\tfrac{5525}{27648}.$$

$$S_5=\frac{1}{1^2.2^2.3^2.4^2.5^2.6^2}+\frac{1}{2^2.3^2.4^2.5^2.6^2.7^2}+\ldots\ \frac{7\pi^2}{2400}-\tfrac{9211}{320000},$$

Etc. Etc. Etc. Etc.

[The above method is due to Mr. W. S. B. Woolhouse, F. R. A. S.]
S_4 is the value required by the problem.

II. Solution by the PROPOSER.

As $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+$, etc., to infinity, equals $\pi^2/6$, we have

$$\frac{1}{1^2.2^2.3^2.4^2.5^2}+\frac{1}{2^2.3^2.4^2.5^2.6^2}+\frac{1}{3^2.4^2.5^2.6^2.7^2}+\ldots.$$

$$=\frac{5.6.7.8}{1^3.2^3.3^3.4^3}\left[\pi^2/6-3\left(\frac{1}{1.2}+\frac{1}{2.3.4}+\frac{1.2}{3.4.5.6.}+\frac{1.2.3}{4.5.6.7.8}\right)\right]$$

$$=\tfrac{35}{288}(\pi^2/6-\tfrac{1105}{672})=.000071559049842333136+.$$

III. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

The series may be thrown into the form

$$\frac{1}{1^2}\cdot\frac{1}{3^2}\cdot\frac{1}{4^2}\cdot\frac{1}{5^2}\left[\left(\frac{1}{1}\right)^2+\left(\frac{1}{6}\right)^2+\left(\frac{1.2}{6.7}\right)^2+\left(\frac{1.2.3}{6.7.8}\right)^2+\left(\frac{1.2.3.4}{6.7.8.9}\right)^2+\ldots.\right]$$

$$=\frac{1}{2^2}\cdot\frac{1}{3^2}\cdot\frac{1}{4^2}\cdot\frac{1}{5^2}\left[1+\frac{1}{6^2}+\frac{1}{\left(\frac{6.7}{1.2}\right)^2}+\frac{1}{\left(\frac{6.7.8}{1.2.3}\right)^2}+\frac{1}{\left(\frac{6.7.8.9}{1.2.3.4}\right)^2}+\ldots.\right]$$

where the denominators are the squares of the successive figurate numbers of the 6th order. Whilst the sum of the reciprocals of the figurate numbers of any order may be expressed by a finite expression, those of the first two orders being excepted since their sums are infinite, the sum of the *squares* of the reciprocals of the figurate numbers cannot so be expressed with the exception of those of the second order, the latter being $\pi^2/6$. Since, however, the series of the giv-

en series converges rapidly, the sum may be found within any limits of precision. A few terms suffice to find the sum correctly within 7 or 8 decimals. The sum of the series without the squares would be exactly $\frac{1}{96}$.

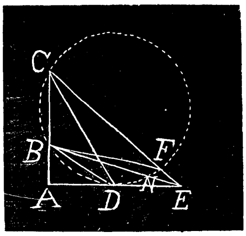
GEOMETRY.

94. Proposed by EDMOND FISH, Hillsboro, Ill.

A tower $AB=a$, is surmounted by a flag pole $BC=b$. A point D is so taken in a line perpendicular to the foot of the tower that angle BDC is a maximum. Prove that AD is a mean proportional between AC and AB.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, East Chester High School, Chester, Pa.; WALTER H. DRANE, A. M., Professor of Mathematics, Jefferson Military College, Washington, Miss.; C. A. JONES, Kosciusko, Miss.; E. R. GIBSON, Wayne, Neb.; and P. S. BERG, Larimore, N. D.

In order that $\angle BDC$ may be a maximum the line AD must be tangent to the circle through B, C at point D. If not let E be some other point in AD, and let EC cut the circle at F.



$\angle BFC > \angle BEC$. But $\angle BFC = \angle BDC$.

$\therefore\ \angle BDC > \angle BEC$. Q. E. D.

$\therefore\ AC : AD = AD : AB$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.; Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Me.; W. L. HARVEY, Portland, Me., and the PROPOSER.

Describing a circle passing through B and C and tangent to AD, the angle BDC is measured by one-half the arc BC, whereas for any other point E the angle BEC is measured by one-half the difference of the arcs BC and FN, therefore by a smaller arc than BC. And since AD is tangent and ABC a secant, we have $AD^2=AC\times AB$.

By the Calculus we may prove this fact thus: Denoting AD by x, we have

$$\tan BDC=\tan(CDA-BDA)=\frac{\tan CDA-\tan BDA}{1+\tan CDA.\tan BDA}$$

$$=\frac{\dfrac{a+b}{x}-\dfrac{a}{x}}{1+\dfrac{a(a+b)}{x^2}}=\frac{bx}{x^2+a(a+b)}.$$

By differentiating, this expression is a maximum for $x^2=a(a+b)$

III. Solution by EDWARD R. ROBINS, Master of Mathematics, Lawrenceville School, Lawrenceville, N. J.: and COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Take A as origin, $AB=a$, $BC=b$, $AD=c$.

Equation of BD is $x/c+y/a=1$.

Equation of CD is $x/c+y/(a+b)=1$.

$$\therefore \measuredangle BCD = \tan^{-1}\frac{-bc}{c^2-a^2-ab},$$

which will be a maximum when $c^2-a^2-ab=0$, or $c^2=a(a+b)$. Q. E. D.

95. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.**

At each point of a parabola is described the rectangular hyperbola of a four-pointic contact; prove that the locus of the center of the hyperbola is an equal parabola.

Solution by the PROPOSER.

The curve having four-pointic contact with the parabola $y^2=4ax$......(1)

is $y^2-4ax-\lambda(yy'-2ax-2ax')^2=0$......(2),

or, $-4\lambda a^2x^2+4a\lambda y'xy+(1-\lambda y'^2)y^2-(4a+8a^2\lambda x')x+4a\lambda x'y'y-4a^2\lambda x'^2=0$..(3).

If this be an equilateral hyperbola,

$4\lambda a^2=1-\lambda y'^2$, or $\lambda=1\div(y'^2+4a^2)$......(4).

Substituting this in (2) and reducing,

$ax^2-y'xy-ay^2+(4a^2+y'^2+2ax')x-x'y'y+ax'^2=0$..... (5).

The center of this is given by

$x=-(y'^2+8a^2)/4a$....(6), $y=y'$....(7). (x', y') being on (1), $y'^2=4ax'$....(8).

Eliminating x', y' from (6), (7), and (8), we have the required locus,

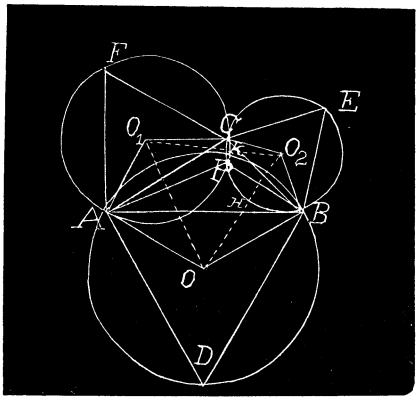
$y^2=-4a(x+2a)$......(9).

96. **Proposed by W. F. BRADBURY, A. M., Head Master, Cambridge Latin School, Cambridge, Mass.**

Isosceles triangles are constructed externally on the three sides of a triangle as bases, with the angles at the bases each 30°. The triangle formed by joining the remote vertices (the 120° vertices) of these isosceles triangles is equilateral. [Geometric—not Trigonometric—solution.]

Solution by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.

The vertices O, O_1, O_2, of the isosceles triangles are the centers of equilateral triangles described on the sides of ABC. Circumferences passed about these triangles intersect in a point, P.



Let P be the intersection of the two circles, AFC and CEB. Join AP, BP, and CP. Since $APCF$ is inscribed, $\angle F+\angle APC=180°$. But $\angle F=60°$.

$\therefore \angle APC=120°$.

Similarly, $\angle CPB=120°$.

$\therefore \angle APB=120°$; and $\angle APB+\angle D=180°$.

$\therefore APBD$ is inscribed, and P is in the circumference of ADB. Q. E. D.

Lines that join the centers of intersecting circles bisect the common

chords and the intercepted arcs. $\therefore$ arc $HP=\frac{1}{2}$ arc AP, and arc $PK=\frac{1}{2}$ arc PC. $\therefore$ arc $HPK=\frac{1}{2}$ arc APC. But arc APC measures $\angle F=60^\circ$ at circumference; therefore HK, its half, measures an equal angle at the center, and $\angle O_2=\angle F=60^\circ$. Similarly, $\angle O_1$ may be shown $=\angle E$, and $\angle O=\angle D$. Since equiangular triangles are equilateral, OO_1O_2 is equilateral. Q. E. D.

Solved in a similar manner by *G. I. HOPKINS*, *F. R. HONEY*, *J. SCHEFFER*, *C. A. JONES*, and *G. B. M. ZERR*.

CALCULUS.

73. Proposed by MOSES COBB STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Ind.

Solve $\int_0^{\frac{1}{4}\pi}\log(1+\tan x)dx$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

$$\int_0^{\frac{1}{4}\pi}\log(1+\tan x)dx=\int_0^{\frac{1}{4}\pi}\log[1+\tan(\tfrac{1}{4}\pi-x)]dx=\int_0^{\frac{1}{4}\pi}\log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$=\int_0^{\frac{1}{4}\pi}\log\left(\frac{2}{1+\tan x}\right)dx=\int_0^{\frac{1}{4}\pi}\log 2.dx-\int_0^{\frac{1}{4}\pi}\log(1+\tan x)dx.$$

$$\therefore 2\int_0^{\frac{1}{4}\pi}\log(1+\tan x)dx=\int_0^{\frac{1}{4}\pi}\log 2.dx=\tfrac{1}{4}\pi\log 2.$$

$$\therefore \int_0^{\frac{1}{4}\pi}\log(1+\tan x)dx=\tfrac{1}{8}\pi\log 2.$$

(See *Todhunter's Integral Calculus*, Art. 51, page 66.)

II. Solution by T. A. CLARK, of the Senior Class, Purdue University, Lafayette, Ind.

$$\int_0^{\frac{1}{4}\pi}\log(1+\tan x)dx=\int_0^{\frac{1}{4}\pi}\log(\tan\tfrac{1}{4}\pi+\tan x)dx=\int_0^{\frac{1}{4}\pi}\log\frac{\sin(\frac{1}{4}\pi+x)}{\cos\frac{1}{4}\pi\cos x}$$

$$=\int_0^{\frac{1}{4}\pi}\log\sec\tfrac{1}{4}\pi dx+\int_0^{\frac{1}{4}\pi}\sin(\tfrac{1}{4}\pi+x)dx-\int_0^{\frac{1}{4}\pi}\log\cos x dx=\tfrac{1}{4}\pi\log\sqrt{2}.$$

To prove $\int_0^{\frac{1}{4}\pi}\log\sin(\frac{1}{4}\pi+x)dx-\int_0^{\frac{1}{4}\pi}\log\cos x dx=0$. Put $\frac{1}{4}\pi+x=\theta$, and

$\frac{1}{2}\pi-x=\phi$, then $\int_0^{\frac{1}{4}\pi}\log\sin(\frac{1}{4}\pi+x)dx=\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\log\sin\theta d\theta$, and $\int_0^{\frac{1}{4}\pi}\log\cos x dx$

$=-\int_{\frac{1}{2}\pi}^{\frac{1}{4}\pi}\sin\phi d\phi=\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\sin\phi d\phi$. Therefore it is evident $\int_0^{\frac{1}{4}\pi}\log\sin(\frac{1}{4}\pi+x)dx$ $-\int_0^{\frac{1}{4}\pi}\cos x dx=0$.

74. Proposed by EDWARD R. ROBBINS, A. B., Mathematical Master in the Lawrenceville School, Lawrenceville, N. J.

A circular ring, whose radii are a and b, is cut by a plane making the area of the section (or sections) a maximum. Required the position of the plane, and the nature and area of the section (or sections).

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

Let the axis of the surface be taken as the axis of z, then the equation to the ring is

$$(x^2+y^2+z^2+ab)^2=(a+b)^2(x^2+y^2).$$

The maximum section will be made by a plane passing through the center of the ring and making an angle β such that $\sin\beta<[(a-b)/(a+b)]$.

Let $z=x\tan\beta$ be this plane.

$\therefore (x^2\sec^2\beta+y^2+ab)^2=(a+b)^2(x^2+y^2)$.

The area of this section is $(\pi/\sec^2\beta)(a^2-b^2)$.

This is a maximum when $\sec^2\beta$ is least, $\therefore \beta=0$.

$\therefore (x^2+y^2+ab)^2=(a+b)^2(x^2+y^2)$.

$\therefore$ the plane coincides with the xy plane and divides the ring into two equal parts. The section is included between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$, area $=\pi(a^2-b^2)$.

MECHANICS.

64. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A cylindrical vessel, radius of vessel r and altitude h, is filled with water and rests on a horizontal plane. It is required to ascertain the maximum angle of elevation to which the plane may be raised without the vessel falling, allowing the coefficient of friction to be such as to prevent sliding, and the water to overflow as the plane is raised.

I. Solution by the PROPOSER.

Let MQ be the inclined plane; $ABCD$ a vertical section of the vessel; $ABFD$ a vertical section of the water when the vessel is on the point of turning over; $AB=2R$, the diameter of the vessel; $AD=H$, its altitude; and the angle $PMN=\theta$, maximum inclination of the plane. The quantity of water in the vessel at the time the vessel is on the point of turning over consists of the cylindrical part whose vertical section is $ABFE$ and the cylindrical ungula whose vertical section is EFD. Let V_2 be the volume of the cylindrical part, and V_1 the volume of the ungula. Let G_2 be the center of gravity of the cylindrical part, G_1 the center of gravity of the ungula, and G_0 the common center of gravity of both.

$AN=AM\cos\beta=r\cos\beta-a\sin\beta$.

Volume of $ADEG=2\pi ar^2$.

Volume of $EGF=2\int_0^{2r}\int_0^{\sqrt{(2rx-x^2)}}\int_0^{x\tan\beta} dxdydz=\pi r^3\tan\beta$.

[This result comes easily without the calculus].

Substituting in equation (1), transposing, dividing through by $\cos\beta$, rearranging, and clearing of fractions,

$$5r^2\tan^3\beta+16ar\tan^2\beta+2(8a^2-3r^2)\tan\beta-16ar=0.$$

Substituting in this, $\frac{1}{2}h-r\tan\beta$ for a, and reducing,

$$5r^2\tan^3\beta-8rh\tan^2\beta+2(5r^2+2h^2)\tan\beta-8rh=0,$$

from which β may be found.

AVERAGE AND PROBABILITY.

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62. **Proposed by O. S. KIBLER, Superintendent of Schools, Middleburg, O.**

A bag contains any number of balls, which are equally likely to be white or black; one is drawn and found to be white. Show that the chance of drawing another white one, the first ball not being replaced, is two-thirds. [From *C. Smith's Treatise on Algebra*, page 615].

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

If m balls are drawn and turn out white, the chance that n others drawn will be white is : $p=[(m+1)/(m+n+1)]$. In the problem, $m=n=1$.

$\therefore p=\frac{2}{3}$. Otherwise, $p=\int_0^1 x^2dx/\int_0^1 xdx=\frac{2}{3}$.

This is the simplest case of the article on page 107, No. 4, Vol. II. of the MONTHLY.

64. **Proposed by REV. W. A. WHITWORTH, A. M.**

O is a given point within a triangle; P is a random point within the same. The line through O and P is produced so as to divide the triangle into a trapezium and a triangle. Find the average area of this triangle. [From the *Educational Times*, London, Eng.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

Let $\triangle$ be the average area required, $\triangle_1$ the average area of BFG; (m, n) coördinates of O. Then the coördinates of E are $(b, bn/m)$; of D, $[b^2n/(ab-am+bn), abn/(ab-am+bn)]$.

Area $ABC=\frac{1}{2}ab\sin C$.

Area $ACO=\frac{1}{2}bn\sin C$.

Area $ADO=bn(am-bn)\sin C/[2(ab-am+bn)]$.

Area $COE=bn(b-m)\sin C/(2m)$.

Area $DOEB=ABC-(ACO+ADO+COE)$.

$\therefore$ Area $DOEB=(a^2b^2m-a^2bm^2+2ab^2mn-abm^2n+b^2n^2m$
$-ab^3n-b^3n^2)\sin C/[2m(ab-am+bn)]$.

The area $DOEB$ does not vary.

The average area of $GOE+DOF=\frac{1}{2}(COE+AOD)$

$$=\frac{(2abm^2n-2b^2mn^2-2ab^2mn+ab^3n+b^3n^2)\sin C}{4m(ab-am+bn)}.$$

$\therefore\ \triangle_1=DOEB+\frac{1}{2}COE+AOD.$

$$\therefore\ \triangle_1=\frac{(2a^2b^2m-2a^2bm^2+2ab^2mn-ab^3n-b^3n^2)\sin C}{4m(ab-am+bn)}.$$

Similarly, $\triangle_2=\dfrac{(2b^2c^2m-2b^2cm^2+2bc^2mn-bc^3n-c^3n^2)\sin C}{4m(cb-bm+cn)}$.

$$\triangle_3=\frac{(2a^2c^2m-2ac^2m^2+2a^2cmn-a^3cn-a^3n^2)\sin C}{4m(ac-cm+an)}.$$

$\triangle=\frac{1}{3}(\triangle_1+\triangle_2+\triangle_3)$. If $a=b=c$ and $m=2n=\frac{2}{3}a$, $\triangle=\frac{1}{8}(a^2\sqrt{3})=\frac{1}{2}$ area of the triangle.

DIOPHANTINE ANALYSIS.

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65. Proposed by MANSFIELD MERRIMAN, C. E., Ph. D., Professor of Civil Engineering, Lehigh University, South Bethlehem, Pa.

Show that the number 1521 can be expressed in seven different ways as the sum of three perfect squares. Can more than seven different ways be found?

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

With the aid of a table that I devised for finding the sum of two squares equal to a number, I found 1521 expressed in *eight* different ways as the sum of three perfect squares. $1521=39^2=2^2+19^2+34^2=2^2+26^2+29^2=9^2+12^2+36^2=10^2+14^2+35^2=13^2+14^2+34^2=13^2+26^2+26^2=14^2+22^2+29^2=19^2+22^2+26^2$.

II. Solution by CHARLES CARROLL CROSS, Libertytown, Md.

The terminal figures of square numbers are, 0, 1, 4, 5, 6 and 9. These may be combined,—taking three at a time, so that the terminal figure will be one in the following manner: (5+5+1)..... (1); (9+1+1)......(2); (1+0+0)..... (3); (9+6+6)......(4); (6+4+1)..... (5); and (6+5+0)...... (6). It readily appears that it is impossible to combine any square numbers represented by (1), (2) and (3) so as to make 1521.

Those of (4) are found to be $13^2+34^2+14^2$ and $13^2+26^2+26^2$; of (5), $34^2+2^2+19^2$, $26^2+22^2+19^2$, $26^2+2^2+29^2$ and $14^2+22^2+29^2$; and of (6), $14^2+36^2+10^2$,—seven in all.

$\therefore 1521=13^2+34^2+14^2=13^2+26^2+26^2=34^2+2^2+19^2=26^2+22^2+19^2$ $=26^2+2^2+29^2=14^2+22^2+29^2=14^2+35^2+10^2$.

III. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Let $1521=m^2$, and x^2, y^2, and z^2 represent the three squares ; then $x^2+y^2+z^2=m^2$, and $x^2=m^2-(y^2+z^2)=m^2-2pm+p^2$. $2pm=p^2+y^2+z^2$. Let $y=tp$ and $z=sp$, then $2m=p(s^2+t^2+1)$. Restoring the value of m, $p=78/(s^2+t^2+1)$, in which s and t may be any rational numbers. Take $s=1$, $t=1$, then $p=26$; $x=m-p=13$, $y=tp=26$ and $z=sp=26$, and $13^2+26^2+26^2=1521$. Take $s=2$, and $t=1$, $p=13$; $x=26$, $y=13$, $z=26$. Take $s=3$, $t=4$, then $p=3$; $x=36$, $y=12$, $z=9$. Take $s=2$, $t=3$, $p=\frac{39}{7}$; $x=\frac{234}{7}$, $y=\frac{78}{7}$, $z=\frac{117}{7}$, and $[(234)^2+(78)^2+(117)^2]/49=1521$.

While this is a solution of the question read *literally*, of course, I understand that the proposer intends to call for integral numbers ; but I have obtained seven integral results, only *by trial.*

Also solved by *SYLVESTER ROBINS*, and *G. B. M. ZERR*.

66. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find two cubic proper fractions whose product is a square proper fraction. Can a *general* solution be made ?

Solution by E. L. SHERWOOD, A. M., Superintendent City Schools, West Point, Miss.; CHARLES CARROLL CROSS, Libertytown, Md.; and G. B. M. ZERR, A. M., Ph. D., Chester High School, Chester, Pa.

Let a^6/b^6, c^6/d^6 be the fractions, $a<b$, $c<d$.

Then $a^6c^6/b^6d^6=(a^3c^3/b^3d^3)^2$.

Let $a=1$, $b=2$, $c=2$, $d=3$.

$\therefore a^6/b^6=\frac{1}{64}$, $c^6/d^6=\frac{64}{729}$;

$\therefore a^6/b^6=(\frac{1}{4})^3$, $c^6/d^6=(\frac{4}{9})^3$; $a^6c^6/b^6d^6=(\frac{1}{27})^2$.

Other fractions can easily be found.

Also solved by *J. H. DRUMMOND*.

67. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Find (1) four consecutive numbers whose sum is a square, and (2) four consecutive numbers the sum of whose squares is a square.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let $x-1$, x, $x+1$ and $x+2$=four consecutive *integers*.

(1) Then their sum$=4x+2=2(2x+1)=2$ *times an odd number*. But this result can never be a square. $\therefore$ The sum of four consecutive integers can *not* be a square.

(2) The sum of their squares$=4x^2+4x+6=2[2(x^2+x+1)+1]=2$ *times an odd number*, which result can never be a square. $\therefore$ The sum of the squares of four consecutive integers can *not* be a square.

If, however, four consecutive numbers may be considered as four *fractions* whose denominators are the same number and equal to 2 *times a square*, and

whose numerators are consecutive integers, then we are able to fulfill the *first part* of the problem.

Of any four consecutive integers we have shown that their sum is 2 *times an odd number*. Now when this odd number is a *square*, we can find four consecutive fractions whose sum is a square, by making the denominators$=2m^2$ and the respective numerators$=2n(n+1)-1$, $2n(n+1)$, $2n(n+1)+1$, and $2n(n+1)+2$. Whence we have $\{[2n(n+1)-1]/2m^2\}+\{[2n(n+1)]/2m^2\}+\{[2n(n+1)+1]/2m^2\}+\{[2n(n+1)+2]/2m^2\}=(2n+1)^2/m^2$.

When $n=m=1$, we find $\frac{3}{2}+\frac{4}{2}+\frac{5}{2}+\frac{6}{2}=3^2$. When $n=1$ and $m=2$, we have $\frac{3}{8}+\frac{4}{8}+\frac{5}{8}+\frac{6}{8}=(\frac{3}{2})^2$. When $n=2$ and $m=1$, we find $\frac{11}{2}+\frac{12}{2}+\frac{13}{2}+\frac{14}{2}=5^2$, etc.

II. Solution by CHARLES CARROLL CROSS, Libertytown, Md.

(1) Combining the consecutive numbers we find that $1+2+3+4$, $5+6+7+8$, $6+7+8+9$, $7+8+9+0$, and $0+1+2+3$ are the only combinations whose terminal figure produces the terminal figure of a square. The first and third combinations can never produce a square number, because a square number whose terminal figure is 0 is always preceded by 0. The second and last combinations cannot produce a square number, because a square number whose terminal figure is 6 is always preceded by an odd number. The fourth combination can never be a square number, because a square number whose terminal figure is 4 is always preceded by an even number. Hence (1) is incorrect.

(2) In the *Mathematical Visitor*, Vol. I, No. 5, page 156, Dr. Martin has shown that the sum of three, of four, and of five consecutive squares, cannot be a square number. Hence (2) is also incorrect. [See also *Mathematical Magazine*, Vol. II, No. 6, page 92.]

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

(1) Let x, $x+1$, $x+2$, $x+3$ be the numbers.

$\therefore 4x+6=a^2$, or $x=(a^2-6)/4$.

$\therefore (a^2-6)/4$, $(a^2-2)/4$, $(a^2+2)/4$, $(a^2+6)/4$ are the numbers.

(2) $4x^2+12x=b^2-14$.

$\therefore x=\frac{1}{2}\sqrt{(a^2-5)}-\frac{3}{2}$, where (a^2-5) must be a square.

Let $a=3$, then $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, are the numbers.

Let $a=2\frac{1}{3}$, then $-\frac{7}{6}$, $-\frac{1}{6}$, $\frac{5}{6}$, $\frac{11}{6}$ are the numbers.

And so for other values of a.

Also solved by *EDWARD R. ROBBINS*, and *J. H. DRUMMOND*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

100. Proposed by CHARLES C. CROSS, Libertytown, Md.

I bought stock at 4% discount, and sold it at $2\frac{1}{2}$% premium, paying a brokerage in both cases of $\frac{1}{4}$%. If my net profits were $130, what was my investment? (Solve by Arithmetic.)

101. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

A man gained $m=3\%$ on his money, in July; and, in August, lost $n=2\%$. What per cent. of his money July 1st is his money September 1st?

⁂ Solutions of these problems should be sent to B. F. Finkel, not later than November 10.

ALGEBRA.

89. Proposed by G. A. MILLER, Ph. D., Instructor in Mathematics, Cornell University, Ithaca, N. Y.

Solve by quadratics.,

$$x^2+y=7\ldots\ldots(1).$$
$$x+y^2=11\ldots\ldots(2).$$

⁂ Solutions of this problem should be sent to J. M. Colaw, not later than November 10.

GEOMETRY.

102. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Ohne Benutzung des Cirkels eine Strecke AC zu halbiren, wenn eine Parallele der Geraden AC gegeben ist. [*Reye's Geometrie der Lage*, Part I, p. 191.]

103. Proposed by FREMONT CRANE, Sand Coulee, Mont.

A horse is tethered with a rope which is attached to a stake B on the edge of a circular pond containing one acre. How long must the rope be to allow the horse to graze over one acre? [From *Home Study Magazine*, problem 249.]

104. Proposed by SAMUEL E. HARWOOD, M. A., Professor of Mathematics, Southern Illinois State Normal University, Carbondale, Ill.

To find a point in the circumference of a semi-circumference such that the sum of its distances from the extremites of the diameter shall be a maximum. [From *Wentworth's Plane Geometry*, Ex. 387.]

⁂ Solutions of these problems should be sent to B. F. Finkel, not later than November 10.

CALCULUS.

80. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A vessel is anchored in three fathoms of water, and the cable passes over a sheave in

the bowsprit, which is six feet above the water. If the cable is hauled in at the rate of a foot a second, how fast is the vessel moving through the water when there is five fathoms of cable out? What is the acceleration of the vessel's velocity? [From *Byerly's Problems in Differential Calculus.*] ANS.—(a) 5/6 feet per second; (b)12/121 feet per second. Are these results correct?

*** Solutions of this problem should be sent to J. M. Colaw, not later than November 10.

MECHANICS.

73. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax Schools, Pittsburg, Pa.

A sixteen-foot plank weighs thirty-two pounds and is supported by two props, four feet and two feet from the ends. What weight is supported by each prop?

74. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

In the experiment of swinging in a vertical circle a glass containing water, and suspended by means of a string, if the string be two feet long, what must be the velocity at the lowest point if the experiment is to succeed? [From *Ziwet's Theoretical Mechanics*, Part III., p. 96.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than November 10.

AVERAGE AND PROBABILITY.

69. Proposed by Rev. W. ALLEN WHITWORTH, M. A.

There are n equal sugar sticks. Each stick is broken into two pieces, all positions of the fracture being equally likely. Of the two n pieces thus formed, a child is to take the largest. Show that his expectation is $[2n+1]/[2(n+1)]$ of a stick. [From *The Educational Times*, June, 1898.]

70. Proposed by Professor MILLER.

A ship at A observes another at B, whose course is unknown. Supposing their speed the same, prove that the chance of their coming within a given distance, d, of each other is always $(2/\pi)\sin^{-1}(d/a)$, whatever the course taken by A; provided its inclination to AB is not greater than $\cos^{-1}(d/a)$, where $AB=a$.

[From *Cambridge Mathematical Tripos*, 1871.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than November 10.

EDITORIALS.

F. W. Hanawalt, formerly of DePauw University, Greencastle, Ind., has been elected Professor of Mathematics in Iowa Wesleyan University, Mount Pleasant, Iowa.

E. M. Shepard, A. M., Professor of Biology in Drury College, is off on leave of absence, visiting the Sandwich Islands, the Philippines, New Guinea, Australia, and Samoa, in the interest of his department.

J. H. Tanner, of Cornell University, and Joseph Allen, of the College of the City of New York, have prepared an Analytical Geometry, which, judging from advanced sheets sent us through the courtesy of the publishers, the American Book Company, promises to be a very excellent work.

Our valued contributor, E. D. Roe, Jr., Associate Professor of Mathematics in Oberlin College, Oberlin, Ohio, had conferred upon him in July the degree of Doctor of Philosophy, *magna cum laude*, by the University of Erlangen, Germany; the subject of his thesis being, *Entwickelung der Sylvester'schen Determinante nach Normal-formen.*

BOOKS AND PERIODICALS.

Elements of Plane and Spherical Trigonometry. By J. W. Nicholson, A. M., LL. D., Professor of Mathematics, Louisiana State University and Agricultural and Mechanical College. 8vo., cloth. Pp. 101+pp. 62 of tables. Price, $1.10. New York and Chicago: The Macmillan Co.

This work presents only the common and most essential elements, it not being the author's aim to write a treatise. It comprises, therefore, what is needed to meet the wants of our best colleges and schools. Among the new features of the book are to be found: the *Trigonometric Circle*, and the introduction of the terms *opposite*, *adjacent* and *like functions*. The mechanical execution of the work bears the "stamp" of the publishers.
B. F. F.

Introduction to Algebra for the Use of Secondary Schools and Technical Colleges. By G. Chrystal, M. A., LL. D., Honorary Fellow of Corpus Christi College, Cambridge; Professor of Mathematics in the University of Edinburgh; Author of "An Elementary Text-Book on Algebra," Two Vols. 8vo. cloth. xxv+412 pp. London: Adam & Chas. Black.

Those who are acquainted with Chrystal's large work on algebra will be interested to examine the present work. Any short notice of it is inadequate to give a clear notion of its merits. There is one criticism that may be offered, viz.: the printed matter is too closely packed, thus marring the appearance of the printed page. B. F. F.

Infinitesimal Analysis. By William Benjamin Smith [Ph. D., Görtingen]. Vol. I. Elementary: Real Variables. Large 8vo. cloth. 352 pages. Price, $3.25. New York: The Macmillan Co.

This volume is in line of excellence with the other works written by Dr. Smith. His *Introductory Modern Geometry of the Point, Ray and Circle*, and his *Co-ordinate Geometry* are books of the highest order, both being written in the light and spirit of modern mathematical teaching. The *Infinitesimal Analysis* treats somewhat in detail the elementary theory and application of the calculus, laying its foundation strong by rigorous argument. The author has given more than usual attention to Hyperbolic Functions, Maxima and Minima, Operators, Tortuous Curves, Partial Derivatives, Multiple Integration, Jacobians, Gamma Functions, etc. The work is one of great interest and value, and will add additional weight to American authorship in this fascinating field of inquiry. It is to be hoped that the great demand for this volume will be an inspiration to its author, and encourage him in the ardious labor of writing Volume II, which, we trust, will soon follow

B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hundson, N. Y.

A first-class magazine, so low in price as to make it possible to be in every home.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.00 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

The *American Monthly Review of Reviews* for October gives special attention to the developments of the past month in international politics and to the lessons of the Spanish-American war. The editor, in the department of "The Progress of the World," discusses the attitude of the Spanish people toward peace conditions, the new relations between Germany and England, the Czar's proposition for disarmament, the Dreyfus case in France, England's reopening of the Soudan. and other serious problems confronting the European powers. Important contributed articles review President McKinley's course in the conduct of the war to a successful close and the deficiencies in our administrative machinery revealed by the fatal delays and break-downs in the medical and subsistence departments of army management.

ERATTA.

June—July Number.

Page 164, line 19, for "$\sqrt{1-n^2}$" read $\sqrt{1+n^2}$.
Page 165, line 9, for "(" read).
Page 168, line 2, for "$(a+cm)^2$" read $(a+cm)^3$.
Page 168, line 12, for "$(c-y)$" read $)c--y)^2$.
Page 168, line 17, for "$\frac{1}{9}\pi$" read $\frac{1}{2}\pi$.
Page 169, line 10, omit].
Page 169, line 12, insert] at end of line.
Page 169, line 15, for "am^2" read a^2m.
Page 169, line 16, for "$(R+r)$" in denominator, read $(R-r)$.
Page 169, line 18, insert] at end of line.
Page 170, line 4 insert R^2 before $\sin^{-1}$.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. OCTOBER, 1898. No. 10.

ON THE GROUPS WHICH ARE DETERMINED BY A GIVEN GROUP.

By DR. G. A. MILLER.

A group (G) generally determines a number of different groups which throw much light on the structure of G. It is the object of this paper to study some of the properties and uses of these groups. When we transform each of the operators of G by any one of them the consequent permutation of these operators determines a substitution. If we use all the operators of any subgroup of G as transformers the corresponding substitutions will evidently form a group which has a 1, α isomorphism to the given subgroup, where the value of α generally depends upon the particular subgroup whose operators have been employed as transformers.

If we employ all the operators of G as transformers the corresponding substitutions form a group which has a 1, β isomorphism to G. This group has been called the *group of cogredient isomorphisms* of G. It is clear that β is the maximum value of α. Hence all the α's are equal to unity whenever $\beta=1$. In this case all the isomorphisms are said to be simple and this process does not lead to any group except those that are found in G.

When some of these isomorphisms are not simple the given process may lead to groups that are not contained in G. The group of lowest order in which this happens is the well known quaternion group of order 8. In this case the group of cogredient isomorphisms is the four-group, which is clearly not contained in the quaternion group. The groups which correspond to the transformations obtained by using the operators of a subgroup are clearly contained in the quaternion group. When the order of such a subgroup exceeds unity $\alpha=\beta=2$.

Since β operators of G must transform its operators according to the same substitution in its group of cogredient isomorphisms, G must contain just β operators that are commutative to each one of its operators. These constitute an Abelian characteristic* subgroup of G. This subgroup may be called the cogredient subgroup of G. Hence the factors of composition of G are the prime factors of β together with the factors of composition of its group of cogredient isomorphisms. The group of cogredient isomorphisms of the group of cogredient isomorphisms is called the second group of cogredient isomorphisms. From the preceding paragraph it follows that the second group of cogredient isomorphisms of the quaternion group is identity.

From the second group of cogredient isomorphisms we may obtain the third in the same manner as the second was obtained from the first, etc. If we arrive at identity by finding the successive groups of cogredient isomorphisms of a given group the group must be solvable since its factors of composition must be prime numbers. There are, however, many solvable groups which do not possess this property, *e. g.* the symmetric group of order 6. Hence the given condition is sufficient but not necessary for the solvability of a group.

G may have simple isomorphisms to itself which cannot be obtained by transforming it by its own operators. All such isomorphisms can be obtained by transforming G by operators that transform it into itself† and they correspond to a group known as the *group of isomorphisms* of G. This group contains the group of cogredient isomorphisms as a selfconjugate (not necessarily characteristic) subgroup. While the group of cogredient isomorphisms of G has a $1, \beta$ isomorphism to G the group of isomorphisms of G need not possess such a property; *e. g.* the group of isomorphisms of the four-group is the symmetric group of order 6.

The group of isomorphisms of G is very useful in the study of groups which contain G as a self-conjugate subgroup. Such a group must transform the operators of G according to its group of isomorphisms or according to some subgroup of this group. In substitution groups the group of isomorphisms is very useful to determine the number of intransitive groups that can be formed by making a group simply isomorphic to itself. Among the types of groups whose groups of isomorphisms have received considerable attention are the Abelian groups which contain no operator, besides identity, whose order differs from a given prime number,‡ the alternating and the symmetric group of any degree,|| and the cyclical group of any order.§

We sometimes arrive at additional groups by finding the successive groups of isomorphisms of G, *i. e.* the group of isomorphisms of the group of isomorphisms, etc. In particular, it is well known that the group of isomorphisms of the group generated by three independent commutative operators of order 2 is the simple group of order 168 and that its second group of isomorphisms is the

*Frobenius, *Berliner Sitzungsberichte*, 1895, page 183.

†Cf. Frobenius, *Berliner Sitzungsberichte*, 1895, page 184.

‡Cf. Moore, *Bulletin of the American Mathematical Society*, vol. 2, 1895, pages 33-43.

||Cf. Hoelder, *Mathematische Annalen*, vol. 46, 1895, pages 333-345.

§Cf. Burnside's *Theory of Groups*, 1897, page 240.

group of order 336, which may be represented as a transitive group of degree 8. We do not obtain any additional group by finding the higher groups of isomorphisms since the group of isomorphisms of a simple group of composite order is necessarily complete.*

It may happen that a group is simply isomorphic to its group of cogredient isomorphisms as well as to its group of isomorphisms. Such a group has been called a complete group. The symmetric group of every degree except two and six is complete. The group of degree 4 and order 8 is an example of a group that is simply isomorphic to its group of isomorphisms without being simply isomorphic to its group of cogredient isomorphisms, while the alternating group of every degree except three is simply isomorphic to its group of cogredient isomorphisms without being simply isomorphic to its group of isomorphisms. The cyclical group of composite order is clearly not simply isomorphic to either of these groups of isomorphisms.

Every selfconjugate subgroup of a group may be regarded as a modulus with respect to which all of its operators may be divided into sets containing an equal number of operators. These sets of operators determine a group known as the quotient group of the given group with respect to the particular selfconjugate subgroup. We have seen that such a quotient group need not be simply isomorphic to any subgroup of the given group.

If s and t represent any two operators of the group (G), $s^{-1}t^{-1}st$ will also represent an operator of G. All the operators of G that can be represented in the form $s^{-1}t^{-1}st$ generate the commutator subgroup of G. This subgroup has also been called the first derivative of G. With respect to this derivative G is isomorphic to an Abelian group of maximum order. The necessary and sufficient condition that G is solvable is that we arrive at identity when we form its successive derivatives.† G may have several self-conjugate subgroups as well as several characteristic subgroups, but it can have only one commutator subgroup as well as only one cogredient subgroup. If a group coincides with its commutator subgroup it is said to be perfect.

If G is represented as a regular group the largest substitution group of the same elements that transforms G into itself determines the holomorph of G,‡ *i. e.* this substitution group is simply isomorphic to the holomorph of G. The subgroup which includes all the substitutions that do not contain any one element of this substitution group determines the group of isomorphisms of G. The holomorph of G may also be defined as the largest group that transforms G into itself and contains only as many operators as are contained in G that are commutative to every operator of G. When G is a complete group the order of its holomorph is the square of the order of G.

When G contains a subgroup that does not include any selfconjugate subgroup besides identity, it can be represented as a transitive substitution group

*Cf. Ibid, page 238.

†Miller, *American Journal of Mathematics*, vol. 20, 1898, page 277.

‡Dyck, *Mathematische Annalen*, vol. 22, 1883, page 90.

whose degree is obtained by dividing the order of G by the order of this subgroup. When the subgroup is a maximum subgroup the corresponding transitive group is primitive; when this condition is not satisfied it is imprimitive*. All subgroups of this type that can be made to correspond in a simple isomorphism of G to itself, and only these, lead to the same transitive group. Hence we can readily obtain all the transitive substitution groups that are determined by G, *i. e.* those which are simply isomorphic to G.

From what precedes it follows that the complete study of a group (G) implies the study of its successive groups of isomorphisms and those of cogredient isomorphisms, its selfconjugate subgroups and the corresponding quotient groups, its successive derivatives and the corresponding solvable groups, its characteristic and cogredient subgroups, its subgroups that do not involve any selfconjugate subgroup besides identity and the corresponding transitive substitution groups, etc. From this standpoint each group generally determines a group complex whose various parts throw much light on the structure of the group.

Chicago, August, 1898.

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*Dyck, *Mathematische Annalen*, vol. 22, 1883, page 90.

INFINITY, THE INFINITESIMAL, AND ZERO.

By HENRY HEATON, M. Sc., Atlantic, Iowa.

So much has been written upon this subject that I do not flatter myself that I can write anything new. I shall only attempt to point out a few of the things that have been written that to my mind plainly cannot be true, and with these errors in view endeavor to express the truth as I see it. If in so doing I fall into error I shall have the satisfaction of knowing that greater men have done the same.

In the *Analyst*, Vol. VIII., pages 105-113, Professor Judson has a very interesting article upon this subject in which he quotes freely from many distinguished authors many things that are plainly fallacious. Yet when he attempts to outline his view of the subject it seems to the writer that he blunders fully as badly as those whom he criticises.

He says: "(7) *A variable which decreases indefinitely and which by reason of its indefiniteness, may be considered as less than any assignable value, is called an infinitesimal.* We shall make use of a horizontal 0 to represent an infinitesimal. Thus, read when $x = \circ$, when x decreases indefinitely or when x is an infinitesimal.

(8) If a be a constant the expression $a/\circ = \infty$ and $a/\infty = \circ$ are rigidly exact."

He says further, "$\infty \pm a = \infty$, also $\circ \times \infty = 0/0$ is wholly indeterminate. We cannot write $a \pm \circ = a$, nor can we write $a \pm \circ = c$, if c is a constant."

He quotes Professor DeMorgan with evident approval as saying that he dates his first clear conception of mathematical infinity from the time when he rejected the relation $a/0 = \infty$.

He says: "Instead of saying the 'tangent is infinite when $x = 90^\circ$' we may say 'the tangent of x becomes infinite as x approaches 90°'. In tracing curves, if $y = x^3/(a-x)$, we should say, y becomes infinite as x approaches a, and not when $x = a$."

Dr. Davies, in his Differential Calculus, Art. 56. says: "When therefore each part becomes *infinitely small*, any *finite* number of them is 0, but an infinite number of them is equal to a finite quantity."

If a *finite* number of them equals 0, how about *one* of them ?

In Lilley's Higher Algebra under the interpretation of $a/0$ I find the following remarkable statement. "Dividing 12 by a number that decreases by 1 each time commencing with $+4$, we have $\frac{12}{+4} = 3$, $\frac{12}{+3} = 4$, $\frac{12}{+2} = 6$, $\frac{12}{+1} = 12$ $\frac{12}{0} = 0$, etc. 3 means that 3 times $+4$ can be subtracted from 12 and leave 0 ; 4 means that 4 times $+3$ can be subtracted from 12 and leave 0; the quotient 0 means that there is *no number of times zero* that the divisor, 0, can be subtracted from 12 and *leave zero*." His general conclusion is then that $a/0 = 0$.

He says: "If a constant be divided by an infinitesimal, the quotient is infinity, and if a constant be divided by infinity, the quotient is infinitesimal." Thus agreeing with Judson.

His conclusion that $a/0 = 0$ is the result of a mere play upon words. He overlooks the fact that if $a/0 = 0$ then if 0 be subtracted from a 0 times the result must be 0.

I am willing to agree with these writers that $a/0$ is the symbol of impossibility ; but I would say that ∞ is also the symbol of impossibility. Hence, $a/0 = \infty$.

a/x does not equal infinity so long as x has the slightest shadow of value.

I believe that mathematical infinity is something beyond the ken of even a Professor DeMorgan, that he was never so far from having a correct conception of it as when he claimed to have a clear conception of it. Tan$x$ does not become infinite as x approaches 90°, for so long as x differs by the shadow of a hair from 90°, tan$x$ is finite.

I think I know exactly what is meant by the term zero. But I can have no conception either of infinity or of the infinitesimal, and I think it would be well if mathematicians would let both pretty severely alone.

If $u = x^2$ and du = an increment that u receives as a result of an increment dx to the value of x ; then $du = 2xdx + dx^2$, and $\frac{du}{dx} = 2x + dx$. This equation cannot reduce to $\frac{du}{dx} = 2x$ in any other way than by putting dx, and consequently

du=absolute zero. If dx has any value at all, infinitesimal or otherwise, $\frac{du}{dx}$ does not equal $2x$.

If $u=x^n$, the general expression for $\frac{du}{dx}$, supposing du and dx to have definite values, is $\frac{du}{dx}=nx^{(n-1)}+\frac{n(n-1)}{1.2}x^{(n-2)}dx+\frac{n(n-1)(n-2)}{1.2.3}x^{(n-3)}dx^2$, etc. When $dx=0$, and not before, this reduces to $\frac{du}{dx}=nx^{(n-1)}$. This is called the differential coefficient of u with respect to x.

It may be ridiculous to say 0/0 equals anything. But there can certainly be no objection to saying that the differential coefficient of a function u with respect to x, is the value to which the *general expression for* $\frac{du}{dx}$ *reduces* when dx is equated to 0, for this is what the differential coefficient really is.

It may be readily shown that the differential coefficients, or derivatives,of two equal functions are always equal, and that if two derivatives are equal *their primitives can differ only by a constant quantity.*

The results obtained by integration are correct because of this last truth.

The process called integration, instead of being the summing of an infinite number of infinitely small quantities, is really the process of finding the primitive corresponding to a given derivative.

As an illustration I shall compute the variable volume, V, of a right cone whose variable altitude is x and the radius of whose variable base is $\frac{ax}{h}$. If dV is any increment which V receives as a result of an increment dx to x, then $dV=\frac{a^2x^2\pi}{h^2}dx+p(dx^2)$. Where p is a variable finite quantity known to be less than the circumference whose radius is $\frac{ax}{h}$. Hence $\frac{dV}{dx}=\frac{a^2x^2\pi}{h^2}+pdx$. When dx is equated to zero the expression for $\frac{dV}{dx}$ reduces to $\frac{a^2x^2\pi}{h^2}$. Hence, $\frac{a^2x^2\pi}{h^2}$ is the derivative of V with respect to x. But $\frac{a^2x^2\pi}{h^2}$ is the derivative of $\frac{a^2x^3\pi}{3h^2}+C$. Hence $V_0{}^r=\frac{a^2h\pi}{3}$.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

98. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A poor man borrowed \$20 which he repaid in eleven monthly installments of \$2 each; what was the annual rate of interest (reckoned as simple interest)?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Natural Science, Chester High School, Chester, Pa.

Here we use the same formula we have used often before in the MONTHLY:

$$p=\frac{Pr(1+r)^n}{(1+r)^n-1}.$$

Where $p=2$, $P=20$, $n=11$.

$\therefore\ 2(1+r)^{11}-2=20r(1+r)^{11}$. $\therefore\ 2(1+r)^{11}(1-10r)=2$.

$\therefore\ r=.016$ nearly. $12r=.192=19\frac{1}{5}\%$ nearly.

$\therefore\ 19\frac{1}{5}\%=$rate of interest.

Also solved by *ELMER SCHUYLER*.

99. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

If 300 cats kill 300 rats in 300 minutes, how many cats will kill 100 rats in 100 minutes?

I. Solution by the PROPOSER.

1. If 300 cats kill 300 rats in 300 minutes,
2. 1 cat will kill 1 rat in 300 minutes,
3. 1 cat will kill 100 rats in 3000 minutes, and
4. 300 cats will kill 100 rats in 100 minutes.

II. Solution by CHARLES C. CROSS, Libertytown, Md.

If 300 cats catch 300 rats in 300 minutes, then 1 cat will catch 300 rats in 9000 minutes, or 1 cat will catch 100 rats in 3000 minutes.

Hence 300 cats will catch 100 rats in 100 minutes.

Also solved by *G. B. M. ZERR*, *FREMONT CRANE*, and *ALOIS F. KAVORIK*.

ALGEBRA.

86. Proposed by J. MARCUS BOORMAN, Consultative Mechanician, and Counsellor at Law, Woodmere, Long Island, N. Y.

Solve $x^2+yz=16\ldots\ldots(A)$; $y^2+xz=17\ldots\ldots(B)$; $z^2+xy=22\ldots\ldots(C)$, for all the roots.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science in Chester High School, Chester, Pa., and Prof. J. SCHEFFER, A. M., Hagerstown, Md.

Let $y=vx$, $z=wx$.

$\therefore\ x^2+vwx^2=16$, $v^2x^2+wx^2=17$, $w^2x^2+vx^2=22\ldots\ldots(1, 2, 3)$.

$(1)\div(2)$, $(1+vw)\div(v^2+w)=\frac{16}{17}$......(4).

$(1)\div(3)$, $(1+vw)\div(w^2+v)=\frac{8}{11}$......(5).

Eliminating v between (4) and (5), we get

$59w^4-378w^3+704w^2-315w-106=0$......(6).

$\therefore$ $w=2$, or $-.21774$, or $2.81226\pm.17307\sqrt{(-1)}$.

$v=1.5$, or 1.02072, or $2.29118\pm.14936\sqrt{(-1)}$.

$x=2$, or 4.53566, or

$y=3$, or 4.62964,

$z=4$, or $-.98759$.

v is found from (5). $\therefore$ $v=(8w^2-11)\div(11w-8)$.

The imaginary values of x, y, and z can be found from $x=4\div\sqrt{(1+vw)}$, $y=vx$, $z=wx$.

[The only rational value of (6) is $w=2$. $\therefore$ $v=\frac{3}{2}$, and substituting $z=2x$, $y=3x\div2$ in the first of the given equations we get $4x^2=16$, whence $x=\pm2$, $y=\pm3$, $z=\pm4$. Dividing the biquadratic by $w-2$, we get the cubic equation $59w^3-260w^2+184w+73=0$, which furnishes the other roots of w, these being, however, irrational. SCHEFFER.]

II. **Solution by the PROPOSER.**

By r (ratio) $y=rx$: ..... $z^2+rx^2=22$......(C).

$x^2+rxz=16$......(A) ; deduct $r(B)-(A)$.

$\therefore$ $r^2x^2+xz=17$......(B) ; $(r^3-1)x^2=17r-15$......(D).

Deduct $r^2(A)-(B)$ ....$(r^3-1)xz=16r^2-17$......(E).

$(r^3-1)(C)$ is......$(r^4-r)x^2+(r^3-1)z^2=22r^3-22$.

$r'(D)$ is...........$(r^4-r)x^2$..........$=17r^2-16r$. Subtracting,

$(r^3-1)z^2=22r^3-17r^2+16r-22$.

Multiplying this equation by (D),

$(r^3-1)^2x^2z^2=374r^3-641r^3+544r^2-630r+352$.

$(r^3-1)^2x^2z^2=256r^4-544r^2+289$. Subtracting $(E)^2$,

$0=118r^4-641r^3+1088r^2-630r+63$ ....$(E)^a$

Obviously $r=1.5$. $\therefore$ 177 $-$ 696 588 $-$ 63

$\therefore$ $118r^3-464r^2+392r-42=0$......(E_1).

$\frac{1}{2}(E_1)$ is..........$59r^3-232r^2+196r-21=0$......(F) $\therefore$ $r=1.5$.

$r_1=0.125069$ 382246 530323 $732\pm$

$r_2=1.021700$ 042125 541767 $819\pm$

$r_a=2.785433$ 965458 436383 025. *Same way*, $z=tx$.

$\therefore$ $59t^3-260t^2+184t+53=0$......(G). *Obviously*, $t=2$.

$t_1=$ 1.207289 829921 583026 $2730\pm$

$t_2=-0.217739$ 956753 32493 .....

$t_a=$ 3.417229 787848 69105...... *Also got by*

$(E)\div(D)=t=(16r^2-17)\div(17r-16)$......($H$) ......is $[16(1.5)^2-17]\div[17(1.5)-16]=2=t$, etc.

Get x by (A)......$x^2(1+rt)=16$; *obviously* $rt=\frac{3}{2}(2)=3$.

$\therefore x^2(1+3)=16$; ......$x=\pm_1 2$; $y=\pm_1 3$; $z=\pm_1 4$.

Test r_2......t_1, etc., by $(B)(C)$ gives $17(t^2+r)=22(r^2+t)$......(J).

Thus $r_1=0.125$......in (J) *fixes* $t_1=1.2073$ (4 true decimals).

$r_2=1.0217$ in (J) *fixes* $t_2=0.217739$ (6 true decimals).

$r_a=2.785434$ in (J) *fixes* $t_a=3.417229$ (6 true decimals).

$\therefore x_1=\pm_1$......to	$y_1=\pm_1$......to	$z_1=\pm_1$ .....
3.728406 657543 8	0.466309 516672 3	4.501267 432220 0
$x_2=\pm_1$......to	$y_2=\pm_1$......to	$z_2=\pm_1$......
4.536281 490681 969	4.634718 990123 0830	0.987729 704967 0174
$x_a=\pm_1$......to	$y_a=\pm_1$......to	$z_a=\pm_1$......
1.233342 644210 0772	3.435894 472231.4779	4.214615 222418.7456

Read $\pm_1$......$\pm_1$......$\pm_1$ "change in unison," not $\pm$ *prime*, $\mp$ *prime*. One page finds *all* 24 roots of x, y, z. Q. V. D.

[*Prof. Charles C. Cross* remarks that two solutions are given in the *Mathematical Diary* for 1831, pages 150-151, and that there is quite an interesting history of this "curious and important question" given by "Unicorn," of North Carolina.

87. Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.

A starts to travel around a circular island at a given point and travels at the rate of 5 miles in 4 hours. One half hour after A, B starts from a point directly opposite from A and travels in an opposite direction at the rate of 4 miles in 3 hours. One hour afterwards C starts from the same point as A and travels in an opposite direction to A at the rate of 3 miles in 2 hours. One half hour afterwards D starts from the same point as B and travels in an opposite direction to B at the rate of 2 miles in 1 hour. Required the size of the island, and when they will all be together, and how far each will have traveled at the accomplishment of this event.

Solution by the PROPOSER.

This problem furnishes a good example in indeterminate analysis. Of course an unlimited number of answers may be obtained each of which will satisfy the conditions of the problem ; but we shall be content to show in what manner one set may be obtained.

A's rate of travel equals $1\frac{1}{4}$ miles per hour, B's rate equals $1\frac{1}{3}$ miles per hour, C's rate equals $1\frac{1}{2}$ miles per hour, and D's rate equals 2 miles per hour. B starts $\frac{1}{2}$ hour after A starts, C starts $1\frac{1}{2}$ hours after A, and D starts 2 hours after A.

Let c=the circumference of the island; then A and B will be together for first time after A starts in $[(\frac{1}{2}c-1\frac{1}{4}\times\frac{1}{2})/(1\frac{1}{4}+1\frac{1}{3})]+\frac{1}{2}=(6c+8)/31$ hours.

For the second time in $[(1\frac{1}{2}c-1\frac{1}{4}\times\frac{1}{2})/(1\frac{1}{4}+1\frac{1}{3})]+\frac{1}{2}=(18c+8)/31$ hours ; and for the $(m+1)$th time in $[(6+12m)c+8]/31$ hours......(A).

A and C will be together for the first time after A starts in $[(c-1\frac{1}{2}\times1\frac{1}{4})/(1\frac{1}{2}+1\frac{1}{4})]+1\frac{1}{2}=(4c+9)/11$ hours.

For the second time in $[(2c-1\frac{1}{2}\times1\frac{1}{4})/(1\frac{1}{2}+1\frac{1}{4})]+1\frac{1}{2}=(8c+9)/11$ hours ; and for the $(n+1)$th time in $[(4+4n)c+9]/11$ hours......(B).

A and D will be together for the first time after A starts in $[(\frac{1}{2}c+1\frac{1}{4}\times 2)/(2-1\frac{1}{4})]+2=(2c+16)/3$ hours.

For the second time in $[(1\frac{1}{2}c+1\frac{1}{4}\times 2)/(2-1\frac{1}{4})]+2=(6c+16)/3$ hours, and for the $p+1$ time in $[(2+4p)c+16]/3$ hours .....(C).

Now when the four persons are all together, the general expressions (A), (B), and (C) will represent the same number of hours. Hence, at the occurrence of such an event, we may equate them, and we shall have

$[(6+12m)c+8]/32=[(4+4n)c+9]/11$......(1), and

$$[(4+4n)c+9]/11=[(2+4p)c+16]/3 \ldots\ldots(2).$$

From equation (1) we obtain $c=191/(132m-124n-68)$......(3).

From equation (2) we obtain $c=149/(12n-44p-10)$......(4).

Equating (3) and (4), clearing of fractions, and dividing the resulting equations by 44, we obtain $447m-472n+191p=153$ .....(5).

Here we have three unknown quantities to be determined from a single equation, but as their values are necessarily restricted to whole numbers, we determine one set of such values as follows: Give to one of the unknown quantities any integral value we choose; as 8 is a convenient value, put $p=8$, and (5) will become $447m-472n=-1375$......(6), or $m=n+(25n-1375)/447$.....(7). Now as the fractional expression $(25n-1375]/447$ must be a whole number, in order to make m such, give to n any integral value in it that shall render the expression a positive quantity. For instance, put $n=70$ and the expression will become $\frac{375}{447}$. We perceive that n must be increased to give an integral result, and by how much we determine as follows:

$(25\times 447-375)/25=332$. $\therefore$ if we take $n=432+70=502$ we shall find that the expression will equal 25, a whole number, and by substitution in (7) we shall find $m=502+25=527$. $\therefore$ the values $p=8$, $m=527$, and $n=502$ will satisfy equation (5). Substituting these values in equations (3) or (4), we have $c=\frac{1}{38}$ of a mile, the circumference of the island. Again, substituting these values found for m, n, p, and c in either of the expressions (A), (B), or (C), we find that A, B, C, and D will be altogether in $5\frac{12}{19}$ hours after A starts.

Proof and remaining answers. In $5\frac{12}{19}$ hours A will travel $\frac{107}{19}\times\frac{5}{4}=7\frac{3}{76}$ miles, and he will go around the island $7\frac{3}{76}\div\frac{1}{38}=262\frac{1}{2}$ times, and hence will be at B's starting point at the end of the time. B, who travels $\frac{1}{2}$ hour less than A, or $5\frac{12}{19}-\frac{1}{2}=\frac{195}{38}$ hours, will travel $\frac{195}{38}\times\frac{4}{3}=6\frac{16}{19}$ miles, and he will go around the island $6\frac{16}{19}\div\frac{1}{38}=260$ times, and hence will be at his starting point at the end of the time. C, who travels $1\frac{1}{2}$ hours less than A, or $5\frac{12}{19}-1\frac{1}{2}=\frac{157}{38}$ hours, will travel $\frac{157}{38}\times\frac{3}{2}=6\frac{15}{76}$ miles, and he will go around the island $6\frac{15}{76}\div\frac{1}{38}=235\frac{1}{2}$ times, and hence will be at B's starting point at the end of the time. D, who travels 2 hours less than A, or $5\frac{12}{19}-2=3\frac{12}{19}$ hours, will travel $3\frac{12}{19}\times 2=7\frac{5}{19}$ miles, and he will go around the island $7\frac{5}{19}\div\frac{1}{38}=276$ times, and he will be at B's starting point at the end of the time. As they will all be at B's starting point at the end of $5\frac{12}{19}$ hours after A starts, they must all be together in that time.

GEOMETRY.

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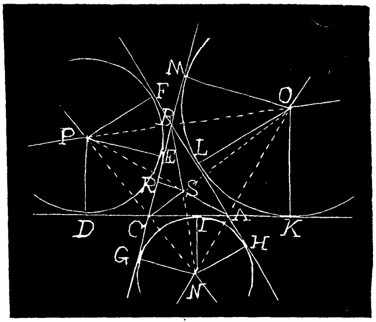
97. Proposed by CHARLES C. CROSS, Libertytown, Md.

Prove by pure geometry: The radius of a circle drawn through the centers of the inscribed and any two escribed circles of a triangle is double the radius of the circumscribed circle of the triangle.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let ABC=any triangle, S be the center of the inscribed circle, and P, N, and O, the centers of the escribed circles. Put $BC=a$, $AC=b$, $AB=c$, $SR=r$, $PE=r_1$, $NH=r_2$, $OM=r_3$, $\dfrac{a+b+c}{2}=s$, $\triangle$=area of $\triangle ABC$.



$$\triangle ABC=\triangle BCS+\triangle ACS+\triangle ABS$$

$$=\frac{ra}{2}+\frac{rb}{2}+\frac{rc}{2}=\tfrac{1}{2}r(a+b+c)=rs \;\ldots\ldots\; (1).$$

Polygon $ACPB=\triangle ABC=\triangle CPB=\triangle ACP+\triangle ABP$; or $rs+\dfrac{r_1a}{2}=\dfrac{r_1b}{2}=\dfrac{r_1c}{2}$; whence $rs=r_1\left(\dfrac{a+b+c}{2}-a\right)$, or $rs=r_1(s-a)$ .... (2).

Polygon $BANC=\triangle ABC+\triangle ANC=\triangle BAN+\triangle BCN$; or $rs+\dfrac{r_2b}{2}=\dfrac{r_2c}{2}=\dfrac{r_2a}{2}$; whence $rs=r_2\left(\dfrac{a+b+c}{2}-b\right)$, or $rs=r_2(s-b)$ .... (3).

Similarly, we find $rs=r_3\left(\dfrac{a+b+c}{2}-c\right)$, or $rs=r_3(s-c)$ .... (4).

Then (1), (2), (3), and (4), respectively, give $rs=\triangle$, $r_1(s-a)=\triangle$, $r_2(s-b)=\triangle$, and $r_3(s-c)=\triangle$. Whence $r=\dfrac{\triangle}{s}$, $r_1=\dfrac{\triangle}{s-a}$, $r_2=\dfrac{\triangle}{s-b}$, and $r_3=\dfrac{\triangle}{s-c}$.

As it is readily proved, we simply state that the lines drawn from the centers of the escribed circles, respectively, to the vertices of the opposite angles of the triangle, are (1) The bisectors respectively of the angles of the triangle, and hence pass through the center of the inscribed circle, and (2) The perpendiculars respectively to the lines joining the centers of the escribed circles; as BN bisects $\angle ABC$ and is perpendicular to PO.

Again, leaving the burden of proof to the reader, hinting, however, that $\angle MBL$ is bisected by PO as a property of escribed circles, we find the following sets of triangles similar: PEB and OLB; OLA and NIA; NIC and PEC.

Let $EC=x$, $IC=y$, $AL=z$; then $BE=a-x$, $AI=b-y$, $BL=c-z$.

From the sets of similar triangles just noted we obtain the following proportions, remembering that $PE=r_1$, $NI=r_2$, and $OL=r_3$.

$c-z : r_3=a-x : r_1$, or $r_1(c-z)=r_3(a-x)$ ....(1).

$z : r_3=b-y : r_2$, or $z=\dfrac{r_3(b-y)}{r_2}$ ....(2).

$y : r_2=x : r_1$, or $x=\dfrac{r_1 y}{r_2}$ ....(3).

Substituting the values of z and x, of (2) and (3), in (1), we find

$$y=\tfrac{1}{2}b+\frac{r_2 a}{2r_1}-\frac{r_2 c}{2r_3}=\tfrac{1}{2}b+\frac{a(s-a)}{2(s-b)}-\frac{c(s-c)}{2(s-b)}=s-a.$$

Whence $b-y=s-c$, $x=s-b$, $a-x=s-c$, $z=s-b$, and $c-z=s-a$.

Then $PB=\sqrt{[(a-x)^2+r_1^2]}=\sqrt{\left(\dfrac{ac(s-c)}{s-a}\right)}$; $BO=\sqrt{\left(\dfrac{ac(s-a)}{s-c}\right)}$; $PO=PB+BO=\dfrac{b\sqrt{(ac)}}{\sqrt{[(s-a)(s-c)]}}$.

Also $AO=\sqrt{\left(\dfrac{bc(s-b)}{s-c}\right)}$; $AN=\sqrt{\left(\dfrac{bc(s-c)}{s-b}\right)}$; $NO=AO+AN=\dfrac{a\sqrt{(bc)}}{\sqrt{[(s-b)(s-c)]}}$.

Also $NC=\sqrt{\left(\dfrac{ab(s-a)}{s-b}\right)}$; $PC=\sqrt{\left(\dfrac{ab(s-b)}{s-a}\right)}$; $NP=NC+PC=\dfrac{c\sqrt{(ab)}}{\sqrt{[(s-a)(s-b)]}}$.

Also $NB=\sqrt{\left(\dfrac{acs}{s-b}\right)}$; $PA=\sqrt{\left(\dfrac{bcs}{s-a}\right)}$; $OC=\sqrt{\left(\dfrac{abs}{s-c}\right)}$.

From the similar triangles PAN and PCS, we have

$PN : PA=PS : PC$, or $\dfrac{c\sqrt{(ab)}}{\sqrt{[(s-a)(s-b)]}} : \sqrt{\left(\dfrac{bcs}{s-a}\right)}=PS : \sqrt{\left(\dfrac{ab(s-b)}{s-a}\right)}$;

whence $PS=\dfrac{a\sqrt{(bc)}}{\sqrt{[s(s-a)]}}$.

Similarly, $OS=\dfrac{c\sqrt{(ab)}}{\sqrt{[s(s-c)]}}$, and $NS=\dfrac{b\sqrt{(ac)}}{\sqrt{[s(s-b)]}}$.

The radius of the circumscribed circle of any triangle equals the product of the three sides divided by four times the area of the triangle.

Put the radius of the circumscribed circle of $\triangle ABC=R$, of $\triangle NSO=R_1$, of $\triangle PSO=R_2$, of $\triangle PSN=R_3$, and of $\triangle PON=R_4$.

Sides of $\triangle ABC$ are a, b, c; and area$=\sqrt{[s(s-a)(s-b)(s-c)]}=\triangle$.

$\therefore R=\dfrac{abc}{4\triangle}$.

Sides of $\triangle NSO$ are $\dfrac{a\sqrt{(bc)}}{\sqrt{[(s-b)(s-c)]}}$, $\dfrac{c\sqrt{(ab)}}{\sqrt{[s(s-c)]}}$, $\dfrac{b\sqrt{(ac)}}{\sqrt{[s(s-b)]}}$; and area $=\dfrac{abc\sqrt{(s-a)}}{2\sqrt{[s(s-b)(s-c)]}}$.

$\therefore R_1=\dfrac{abc}{2\triangle}=2R.$

Sides of $\triangle PSO$ are $\dfrac{b\sqrt{(ac)}}{\sqrt{[(s-a)(s-c)]}}$, $\dfrac{c\sqrt{(ab)}}{\sqrt{[s(s-c)]}}$, $\dfrac{a\sqrt{(bc)}}{\sqrt{[s(s-a)]}}$; and area $=\dfrac{abc\sqrt{(s-b)}}{2\sqrt{[s(s-a)(s-c)]}}$.

$\therefore R_2=\dfrac{abc}{2\triangle}=2R.$

Sides of $\triangle PSN$ are $\dfrac{c\sqrt{(ab)}}{\sqrt{[(s-a)(s-b)]}}$, $\dfrac{a\sqrt{(bc)}}{\sqrt{[s(s-a)]}}$, $\dfrac{b\sqrt{(ac)}}{\sqrt{[s(s-b)]}}$; and area $=\dfrac{abc\sqrt{(s-c)}}{2\sqrt{[s(s-a)(s-b)]}}$.

$\therefore R_3=\dfrac{abc}{2\triangle}=2R.$

Sides of $\triangle PON$ are $\dfrac{b\sqrt{(ac)}}{\sqrt{[(s-a)(s-c)]}}$, $\dfrac{a\sqrt{(bc)}}{\sqrt{[(s-b)(s-c)]}}$, $\dfrac{c\sqrt{(ab)}}{\sqrt{[(s-a)(s-b)]}}$, and area $=\dfrac{abc\sqrt{s}}{2\sqrt{[(s-a)(s-b)(s-c)]}}$.

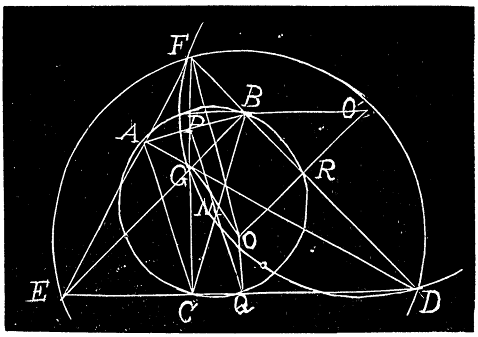
$\therefore R_4=\dfrac{abc}{2\triangle}=2R.$

Hence the general proposition: The centers of the escribed and inscribed circles of a triangle are the vertices of four different triangles the radii of whose circumscribed circles are equal.

Take $a=14$, $b=13$, $c=15$. Then $r=4$, $r_1=12$, $r_2=10\frac{1}{2}$, $r_3=14$, and $R=8\frac{1}{8}$.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

Let ABC be the given triangle, then if G is the incenter of ABC, E, F, D are its excenters. The circumcircle of ABC is the nine-points circle of EDF and G its orthocenter. Hence PQ is a diameter of the nine-points circle, since P is the mid-point of FG and Q the mid-point of ED.



$\therefore$ The mid-point of PQ is the center of the nine-points circle.

Let O be the circumcenter. Then the mid-point of GO is the center of the nine-points circle; since perpendiculars to the mid-points of CQ and BR, chords of the nine-points circle, intersect at the mid-point of GO and also at the center of the nine-points circle.

$\therefore$ PQ and GO bisect one another at M. From the triangles MQO and GPM, $GM=OM$, $PM=QM$, $\angle OMQ=\angle PMG$. $\therefore$ $OQ=PG=FP$, and OQ is also parallel to FP. $\therefore$ $OF=PQ$.

$\therefore$ The radius of EFD=the diameter of ABC. Perpendicular to FG at the point P draw $PO'=QD$. Then since $FP=PG=OQ$, $O'F=O'G=OD$.

$\therefore$ $OF=O'F=OD=O'D=O'G$.

$\therefore$ radius of FGD=radius EFD=diameter ABC.

Similarly radius EGD and radius EGF=radius (each) EFD.

98. **Proposed by EDWARD R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School, Lawrenceville. N. J.**

Construct a circle which shall pass through two given points and touch a given circle, (1) when the distance between the points is less than the diameter of the circle, and (2) when it is greater.

I. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

The construction for both cases is the same.

I Case. Let A, B be the given points, $CMDK$ the given circle. Through A, B draw the circle $ABEF$ intersecting the given circle in E, F. Draw AB, EF intersecting at G. Draw the tangents GK, GM. Draw QQ_1 perpendicular to AB at its mid-point. Through R, the center of $CMDK$, draw RM, RK intersecting QQ_1 in O_1, O. Then O, O_1 are the centers of two circles satisfying the conditions, and ABK, ABM are the circles.

II Case. Let C, D be the given points, $AHBL$ the given circle. Through C, D describe the circle $CDEF$ intersecting the given circle in E, F. Draw CD, EF intersecting at G. Draw the tangents GH, GL. Draw RR_1 perpendicular to CD at its mid-point. Through Q, the center of $AHBL$, draw QL, QH intersecting RR_1 in P, P_1. Then P, P_1 are the centers of two circles satisfying the conditions, and CDL, CDH are the circles.

In the above both points are without the given circle. This problem is thoroughly discussed on page 271, No. 8, Vol. I., THE AMERICAN MATHEMATICAL MONTHLY.

II. **Solution by FREDERIC R. HONEY, Ph. B., New Haven, Conn.**

The following description applies when the distance between the points is less, and when greater than the diameter of the circle.

Let a and b be the given points and A the circumference of the given circle.

Through a and b pass a circle the circumference of which intersects A at c and d. Draw ba and dc and produce these lines until they meet at e. Draw ef tangent to A. Through the point of tangency f and the given points a and b pass the required circle C. Since two tangents may be drawn there are, in each case, two solutions.

Analysis of the construction : $eb\times ea=ed\times ec=(ef)^2$.

[NOTE. For a demonstration of this same proposition with a diagram, see Vol. I., page 271. Professors Zerr and Honey each furnished neat diagrams with these demonstrations, but we believe the demonstrations sufficiently clear without them. ED. F.]

CALCULUS.

75. Proposed by O. W. ANTHONY, M. Sc., Instructor in Mathematics, Boys' High School, New York City.

Solve the differential equation $\dfrac{d^2y}{dx^2}+n^2y=\dfrac{6}{x^2}y$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $y=ux^3$, the equation then becomes $\dfrac{d^2u}{dx^2}+\dfrac{6}{x}.\dfrac{du}{dx}+n^2u=0$.

Consider the equation $d^2v/dx^2+n^2v=0$.

Then $(dv/dx)^2+n^2v^2=A=n^2c^2$.

$\therefore\ \pm dv/\sqrt{(C^2-v^2)}=ndx$. $\therefore\ v=c\cos(nx+\alpha)$, or $c\sin(nx+\alpha)$.

Let $x^2=dz$, and change the independent variable from x to z.

$\therefore\ 2z(d^2v/dz^2)+dv/dz+n^2v=0$.

Differentiate this last three times with regard to z, and let $t=d^3v/dx^3$, then we get $2z.(d^2t/dz^2)+7.(dt/dz)+n^2t=0$.

Now re-change the independent variable from z to x.

$$\therefore\ d^2t/dx^2+\frac{6}{x}.\frac{dt}{dx}+n^2t=0.$$

$$\therefore\ u=t=\frac{d^3v}{dz^3}=\left(\frac{1}{x}.\frac{d}{dx}\right)^3[c\cos(nx+\alpha)].$$

$$\therefore\ y=x^3\left(\frac{1}{x}.\frac{d}{dx}\right)[c\cos(nx+\alpha)],$$

$$=cn^3\left[\left(1-\frac{3}{n^2x^3}\right)\sin(nx+\alpha)+\frac{3}{nx}\cos(nx+\alpha)\right],$$

$$=C\left[\left(1-\frac{3}{n^2x^2}\right)\sin(nx+\alpha)+\frac{3}{nx}\cos(nx+\alpha)\right].$$

If we had written $v=c\sin(nx+\alpha)$,

$$y=C\left[\frac{3}{nx}\sin(nx+\alpha)-\left(1-\frac{3}{n^2x^2}\right)\cos(nx+\alpha)\right].$$

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

This is a "case" of Riccati's Equation of the form

$$\frac{d^2u}{dx^2}-a^2x^{2q-2}u=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and of scientific interest as it appears in the theory of the figure of the earth. See Airy's *Figure of the Earth*, articles 64, 65; Col. Clark's *Geodesy*, page 80; Tisserrand's *Méc. Cel.*, *Tome* 2, Chapitre xiv *Equilibre d'Une Masse Fluide Hétérogène*, in the last of which the equation is

$$\frac{d^2y}{dx^2}=\left\{\frac{a^2\dfrac{d\rho}{da}}{\int_0^a \rho a^2 da}+\frac{6}{a^2}\right\}y\ \ldots\ldots\ldots\ldots\ldots\ldots..(2).$$

It is due to this connection, that the proposed equation has received a variety of solutions.

What is here given is based on a transformation of (1) and a pair of solutions given in Johnson's *Differential Equations*, articles 213, 214.

(1) is transformed into

$$\frac{d^2u}{dx^2}-\frac{m-1}{x}\frac{du}{dx}-a^2u=0\ \ldots\ldots\ldots\ldots\ldots\ldots..(3),$$

two integrals of which are of the form

$$u_3=e^{ax}\left(1-\frac{m-1}{m-1}ax+\frac{(m-1)(m-3)a^2x^2}{(m-1)(m-2)\,2\,!}-\ldots.\right)\ \ldots\ldots\ldots.(4),$$

$$u_5=e^{-ax}\left(1+\frac{m-1}{m-1}ax+\frac{(m-1)(m-3)a^2x^2}{(m-1)(m-2)\,2\,!}+\ldots.\right)\ \ldots\ldots\ldots.(5),$$

and connected with u by the relation $u=Au_3+Bu_5$ ....(6).

Now assuming $u=x^pv$.........(7), or $v=ux^{-p}$.........(8), (3) becomes

$$\frac{d^2v}{dx^2}-a^2v=\frac{p(p+1)v}{x^2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Putting $a=n\surd(-1)$, $p=2$, $m=2p+1=5$, and $v=y$, $\surd(-1)=i$, (6) and (8), with (4) and (5) give

$$y=ux^{-2}=x^{-2}[Ae^{nix}+Be^{-nix}-\tfrac{1}{3}n^2x^2(Ae^{nix}+Be^{-nix})+nx(Bie^{-nix}-Aie^{nix})]\ \ldots(10),$$

which it is not difficult to reduce to the form,

$$y=Cx^{-2}[(3-n^2x^2)\cos(nx+\alpha)+3nx\sin(nx+\alpha)]\ldots\ldots\ldots..(11).$$

See Forsyth's *Differential Equations*, Ed. 1885, Ex. 3, page 65; Ex. 1, page 175; Ex. 21, page 180; Ex. 26, page 181, and Ex. 5, pages 233, 4.

76. Proposed by E. B. ESCOTT, Cambridge, Mass.

Solve the partial differential equation, $q^2r+4pqs+p^2t+p^2q^2(rt-s^2)=a^2$. [*Forsyth's Differential Equations*, page 376.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The equation in λ is, $\lambda^2p^2q^2(1+a^2)+4\lambda p^3q^3+p^4q^4=0$.

$\therefore\ \lambda^2(1+a^2)+4\lambda pq+p^2q^2=0$.

$\therefore\ \lambda=-\{pq/(1+a^2)\}[1\mp\surd(3-a^2)]=m_1pq$ or m_2pq.

The first system of integrals is $p^2q^2dy+m_1p^3qdx+m_1p^3q^3dp=0$.

$p^2q^2dx+m_2pq^3dy+m_2p^3q^3dq=0$, or $qdy+m_1pdx+m_1pq^2dp=0$..........(1);

$pdx+m_2qdy+m_2p^2qdq=0$ ....(2).

Similarly for second system $pdx+m_1qdy+m_1p^2qdq=0$............(3);

$qdy+m_2pdx+m_2pq^2dp=0$ ....(4).

(1)+(3) gives $(pdx+qdy)(1+m_1)+m_1(pq^2dp+p^2qdq)=0$, but $dz=pdx+qdy$.

$\therefore (1+m_1)dz+m_1)pq^2dp+p^2qdq)=0$.

$\therefore 2z(1+m_1)+m_1p^2q^2=2a$ ....(5).

Similarly (2)+(3) gives $2z(1+m_2)+m_2p^2q^2=2b$..................(6).

Eliminating p^2q^2 between (5) and (6) we get $z(m_1-m_2)=bm_1-am_2$.

77. Proposed by T. E. COLE, Columbus, Ohio.

Derive the equation of a point in a pedal of a bicycle as the wheel rolls along on a plane.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; and A. E. BREECE, A. B., Professor of Mathematics, Portland University, Portland, Oregon.

Let CO be the radius of a circle whose circumference is equal in length to the distance C moves in one revolution of the pedal. P a point in the pedal, A, A' points where CP is perpendicular to the fixed line AA'.

Let $CO=a$, $CP=d$, $\angle OCP=\theta$, then we have

$$x=AN=AO-ON=a\theta-d\sin\theta\,;\ y=PN=CO+PM=a-d\cos\theta.$$

$\therefore x=a\cos^{-1}[(a-y)/d]-\sqrt{[d^2-(a-y)^2]}$.

If $a=d$, the curve is a right cycloid.

If $a<d$, the curve is a prolate cycloid.

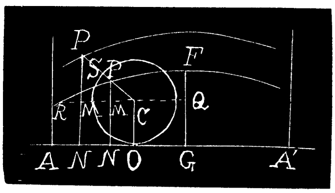
If $a>d$, the curve is a curtate cycloid.

II. Solution by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military College, Washington, Miss.

Consider the curve traced by the pedal in one turn. Take origin at the lowest point of the pedal's path. Let axis of y be vertical and axis of x the straight line joining any two consecutive lowest points of pedal's path. Take P any point in the curve and draw its ordinate PN. Let F be the middle point of curve and FG its ordinate. From Q, the middle point of FG, draw QR parallel to the x-axis. Take $PC=FQ$ and draw CO parallel to PN.

Now let a be length of pedal arm, r the radius of bicycle wheel, n the number of turns the wheel makes to one turn of the pedal, and θ the angle PCO through which the pedal arm has turned on reaching the point P in the curve.

Then from the figure, we have,



$$x=AN=AG-(GO+MC)=\pi rn-\left[\left(\pi rn-\frac{\pi rn\theta}{180}\right)+a\sin\theta\right]=\frac{\pi rn}{180}\theta-a\sin\theta.$$

$$y=PN=MN+MP=a-a\cos\theta=a\operatorname{vers}\theta.$$

Whence the equation of curve is, $x=\dfrac{\pi rn}{180}\operatorname{vers}^{-1}\dfrac{y}{a}-\sqrt{(2ay-y^2)}$.

MECHANICS.

66. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The distance, parallel to the axis, from the mid-point of a chord to the arc of a parabola is constant. Show that the center of gravity of all segments formed by the chord is an equal parabola.

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University, Miss.

Taking the tangent parallel to the chord as y-axis, and the corresponding diameter as x-axis, denoting the inclination of the axes by β, and the latus rectum by $2p$, the equation of the parabola is

$$y^2=\frac{2p}{\sin^2\beta}x.$$

Since all chords parallel to the base of the segment are bisected by the x-axis, the center of gravity is on this axis.

$\bar{x}$ being the abscissa of the center of gravity and a the distance from the origin to the mid-point of the chord, we have, by taking moments about the tangent,

$$\bar{x}\sin\beta=\frac{\int_0^a y\sin\beta dx.x\sin\beta}{\int_0^a y\sin\beta dx},\text{ or }\bar{x}=\frac{\int_0^a \sqrt{(2p)}x^{\frac{3}{2}}\,dx}{\int_0^a \sqrt{(2p)}x^{\frac{1}{2}}\,dx}=\tfrac{3}{5}a.$$

Since a is constant for all positions of the chord, if the given parabola be moved to the right a distance equal to $\frac{3}{5}a$, it will coincide with the locus of the centers of gravity. Hence this locus is an equal parabola.

67. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A conical stick of timber, length a, radius of base r, and density δ, is depressed, apex downward, in a liquid, density δ', so that the base is just level with the liquid. If left free to rise, required the greatest altitude to which it will ascend.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.

Let x=the part of the axis above the liquid at any time t from the beginning of the motion; then the volume of the cone being $\frac{1}{3}\pi r^2a$, the part under the liquid is $\frac{1}{3}\pi r^2a[(a-x)/a]^3$, and if ρ be the density of the cone, the force acting on the cone is proportional to the members of the equation

$$\tfrac{1}{3}\pi r^2a\rho\frac{vdv}{dx}=g[\tfrac{1}{3}\pi r^2a\delta'\left(\frac{a-x}{a}\right)^3-\tfrac{1}{3}\pi r^2a\rho]\ \ldots\ldots\ldots\ldots(1),$$

v being the velocity.

Or (1) is $a^3\rho\dfrac{vdv}{dx}=g[\delta'(a-x)^3-\rho a^3]$ ....(2).

Integrating (2), $\frac{1}{2}a^3\rho v^2=g[-\frac{1}{4}\delta'(a-x)^4-\rho a^3x]+C$..................(3).

When $x=0$, $v=0$, $\therefore$ $C=\frac{1}{4}(\delta' a^4 g)$ and (3) becomes

$$\tfrac{1}{2}(a^3\rho)v^2=g\{(\tfrac{1}{4}\delta')[a^4-(a-x)^4]-\rho a^3x\}. \quad\ldots\ldots\ldots\ldots(4).$$

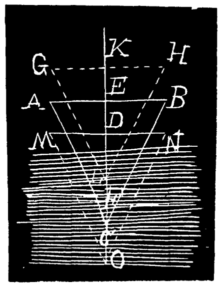
At the greatest height, $v=0$, and then

$$(\tfrac{1}{4}\delta')[a^4-(a-x)^4]-\rho a^3x=0 \quad\ldots\ldots\ldots\ldots(5),$$

a biquadratic for x. One value plainly being 0, there remains a cubic to be solved.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

Let $DK=x$, $DL=y$, $\therefore$ $x+y=a$. When the cone is under the liquid its acceleration is $g(\delta'-\delta)/\delta$.



The equation of motion is $d^2x/dt^2=\beta y^3-g$.

When $y=a$, $d^2x/dt^2=g(\delta'-\delta)/\delta$.

$\therefore$ $\beta=\delta' g/a^3\delta$.

$\therefore$ $d^2x/dt^2=(g/a^3\delta)(\delta' y^3-\delta a^3)=(g/a^3\delta)[\delta'(a-x)^3-\delta a^3]$.

$\therefore$ $(dx/dt)^2=-(g/2a^3\delta)[(a-x)^4\delta'+4a^3x\delta]+A$.

When $y=a$, $dx/dt=0$, $x=0$.

$\therefore$ $A=ag\delta'/2\delta$.

$\therefore$ $(dx/dt)^2=v^2=(ag\delta'/2\delta)-(g/2a^3\delta)[\delta'(a-x)^4+4\delta a^3x]$.

When $v=0$, $\delta'(a-x)^4+4\delta a^3x=a^4\delta'$, or

$$x^3-4ax^2+6a^2x=4a^3(\delta'-\delta)/\delta'.$$

This cubic determines x. When $\delta<\frac{1}{4}\delta'$, the cone leaps out of the liquid. At this moment $x=a$, $\therefore$ $v^2=ag(\delta'-4\delta)/2\delta$.

Then $h=v^2/2g=a(\delta'-4\delta)/4\delta=$height of vertex above water.

$\therefore$ Total height of base above liquid$=a+h$.

68. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

Find the horizontal and vertical components of the moon's "disturbing force" for any point on the earth's surface making an angle φ with the line joining the center of the earth to the center of the moon.

Solution by the PROPOSER.

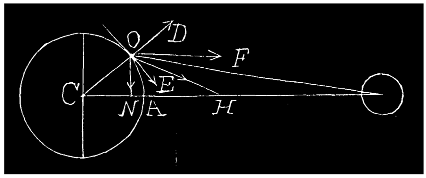
Let CM, the distance from the earth to the moon$=d$; CA, the radius of the earth$=a$; O the point on the earth; $\angle OCA=\varphi$; m=moon's mass; $f=$ "moon's disturbing force" in direction MN; f'=disturbing force in direction ON.

Then $f'=(m/OM^2)\times(ON/OM)=m(ON/OM^3)$.

$ON=a\sin\varphi$, and we can take $OM=d$ without any appreciable error.

$$\therefore f' = \frac{ma\sin\varphi}{d^3} = Q\sin\varphi,$$

where $Q = ma/d^3$.



$$f = m\left(\frac{1}{NM^2} - \frac{1}{CM^2}\right) = m\left(\frac{CM^2 - NM^2}{CM^2.NM^2}\right)$$

$$= \frac{mCN(CM+NM)}{CM^2.NM^2} = \frac{m.CN(2d-CN)}{d^2(d-CN)^2}$$

$$= \frac{2m.CN}{d^3} \cdot \frac{(1-CN/2d)}{(1-CN/d)^2}.$$

Since CN/d is small we may neglect it

$$\therefore f = \frac{2m.CN}{d^3} = \frac{2ma\cos\varphi}{d^3} = 2\,Q\cos\varphi.$$

Now let f=horizontal, f'=vertical components.

$$\therefore f = f\sin\varphi + f'\cos\varphi = 3Q\sin\varphi\cos\varphi = \tfrac{3}{2}Q\sin2\varphi.$$

$$f' = f\cos\varphi - f'\sin\varphi = Q(2\cos^2\varphi - \sin^2\phi) = Q(3\cos^2\varphi - 1).$$

DIOPHANTINE ANALYSIS.

68. **Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.**

Find a *general* value for p in the expression $4p+1$=the sum of two squares.

I. **Solution by J. H. DRUMMOND, LL. D., Portland, Me., and G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.**

Since $4p+1$ is odd one of the squares must be even and the other odd. Take $2q+1$ as one of the numbers and $2s$ for the other, and we have $4q^2+4q+1+4s^2=4p+1$. Hence $p=q^2+q+s^2$ in which q may be zero or any number; and s any number.

II. **Solution by the PROPOSER.**

From a table in which I have all the odd numbers, up to 12013, that are equivalent to the sum of two squares, I find by inspection that all the values for $4p+1$ can be obtained by making $p=\frac{n(n+1)}{2}+\frac{a(a+1)}{2}$, in which $n>a$.

Whence $4\left[\frac{n(n+1)}{2}+\frac{a(a+1)}{2}\right]+1=(n+a+1)^2+(n-a)^2$. (I).

When $a=0$, we have $4\left[\frac{n(n+1)}{2}\right]+1=(n+1)^2+n^2$; and we obtain the series of values 5, 13, 25, 41, 61, 85, etc.

When $a=1$, we have $4\left[\frac{n(n+1)}{2}+1\right]+1=(n+2)^2+(n-1)^2$; and we obtain the series of values 17, 29, 45, 65, 89, 117, etc.

When $a=2$, we have $4\left[\frac{n(n+1)}{2}+3\right]+1=(n+3)^2+(n-2)^2$; and we obtain the series of values 37, 53, 73, 97, 125, 157, etc.

We observe that the development of $4p+1$ occurs in *series* ; the *number of terms in each series* as well as the *number of series* being infinite.

We also notice that the *number of series*$=a+1$. Now put $r=n-a$. We then readily deduce that the rth *term of any series*

$$=4\left[\frac{(r+a)(r+a+1)}{2}+\frac{a(a+1)}{2}\right]+1=(r+2a+1)^2+r^2. \quad \text{(II)}.$$

For the numerical development of $4p+1$, Formula II is better adapted than Formula I, as the values of r and a are independent of each other. But we can still improve a little in this direction. Put $N=a+1=$*the number of the series.*

Then (A), the rth term of the Nth series, and also, (B), the Nth term of the rth series

$$=4\left[\frac{(N+r)(N+r-1)}{2}+\frac{N(N-1)}{2}\right]+1=4\left[N(N+r-1)+\frac{r(r-1)}{2}\right]+1$$

$$=(2N+r-1)^2+r^2. \quad \text{(III)}.$$

From this we observe that there are *two forms of series* embodied in one formula, each form, however containing all the values of $4p+1$.

The one form, of which we have already treated and in which $r=$the consecutive integers for each value of N, or for each series, we shall designate as "Series A."

The other form, in which $N=$the consecutive integers for each value of r, or for each series, we shall term "Series B."

The *first series* of "Series B" consists of the *first terms* of the consecutive series of "Series A"; as 5, 17, 37, 65, 101, 145, etc.

The *second series* of "Series B" consists of the second terms of the consecutive series of "Series A"; as 13, 29, 53, 85, 125, 173, etc.; and so on for the respective series.

Two other values of p, in Formula III, are

$$\frac{(2N+r)(r-1)}{2}+N^2 \text{ and } (N-1)(N+r)+\frac{r(r+1)}{2},$$

obtained by different arrangement of terms and factoring. The two values are significant ; as $4\left[\frac{(2N+r)(r-1)}{2}\right]$ is the difference between the rth and first terms in "Series A," and $4(N-1)(N+r)$ is the difference between the Nth and first terms of "Series B."

We have also a general formula for finding *consecutively* the terms of a series:—

In "Series A", rth term $+4(N+r)=r+1$th term.

In "Series B", Nth term $+4(2N+r)=N+1$th term.

The following mathematical diversions may be interesting as bearing upon this problem.

Knowing the first series in "Series A", we can find the other series consecutively by means of the following rule: *Add to each term of the last found series, omitting the first term, the number of the series times* 4, or $4N$; as

First series.	5,	13,	25,	41,	61,	85,	113, etc.
Add 1×4		4,	4,	4,	4,	4,	4,
Second series		17,	29,	45,	65,	89,	117, etc.
Add 2×4			8,	8,	8,	8,	8,
Third series			37,	53,	73,	97,	125, etc., etc.

The number of the series $=N=\frac{1}{2}\sqrt{(1\text{st term}-1)}$; or $4N^2+1=$1st term.

In "Series B", $4\left[\frac{r(r+1)}{2}\right]+1=$1st term.

The following is a most interesting deduction from Formula III.

(1), $(2N+r-1)^2+r^2=(2N-1)(2N+2r-1)+2r^2=2r(2N+r-1)+(2N-1)^2$,

(2), $[(2N-1)(2N+2r-1)]^2+[2r(2N+r-1)]^2=[(2N+r-1)^2+r^2]^2$, or $=$ the *square of* $4p+1=$the sum of two squares.

The application of these deductions gives rise to the annexed table, which I have constructed and extended to over three thousand numbers each equal to $4p+1=$the sum of two squares. It contains every number of the kind up to $12013=77^2+78^2$. By means of the table we can readily find—

(1). The two numbers the sum of whose squares$=$the given numerical value of $4p+1$.

(2). The two numbers the sum of whose squares$=$the square of the given numerical value of $4p+1$.

The *first column* consists of the consecutive values of r, and is called the "r column" of consecutive integers.

The other columns are the consecutive series of "Series A."

The *first row* is the "N row" of consecutive integers.

The *second row* is the "$2N-1$ row" of consecutive odd numbers.

The other rows are the consecutive series of "Series B."

The number of the column of values of $4p+1$ is indicated at its top by the value of N; and the number of the row of values is shown at its left by the value of r.

In using the table, all mention of values of r, N, and $2N-1$, refer to the respective values in the *same row* and the *same column* in which is found the given value of $4p+1$.

To find the two numbers the sum of whose squares$=4p+1$; one of the numbers$=r+2N-1$ and the other number$=r$. Take 97, at the intersection

of *column* 3 with *row* 4. Then $4+5=9$=one number, and 4=the other number; $97=9^2+4^2$.

To find the two numbers the sum of whose squares=the square of $4p+1$; one of the numbers$=4p+1-2r^2$, and the other number$=4p+1-(2N-1)^2$.

Take 97. Then $97-2\times4^2=65$=one of the numbers, and $97-5^2=72$; $97^2=65^2+72^2$.

TABLE OF VALUES OF $4p+1$=THE SUM OF TWO SQUARES.

	1	2	3	4	5	6	7	8	9	10	11	12	13
	1	3	5	7	9	11	13	15	17	19	21	23	25
1	5	17	37	65	101	145	197	257	325	401	485	577	677
2	13	29	53	85	125	173	229	293	365	445	533	629	733
3	25	45	73	109	153	205	265	333	409	493	585	685	793
4	41	65	97	137	185	241	305	377	457	545	641	745	857
5	61	89	125	169	221	281	349	425	509	601	701	809	925
6	85	117	157	205	261	325	397	477	565	661	765	877	997
7	113	149	193	245	305	373	749	533	625	725	833	949	1073
8	145	185	233	289	353	425	505	593	689	793	905	1025	1153
9	181	225	277	337	405	481	565	657	757	865	981	1105	1237
10	221	269	325	389	461	541	629	725	829	941	1061	1189	1325
11	265	317	377	445	521	605	697	797	905	1021	1145	1277	1417
12	313	369	433	505	585	673	769	873	985	1105	1233	1369	1513
13	365	425	493	569	653	745	845	953	1069	1193	1325	1465	1613
14	421	485	557	637	725	821	925	1037	1157	1285	1421	1565	1717
15	481	549	625	709	801	901	1009	1125	1249	1381	1521	1669	1825
16	545	617	697	785	881	985	1097	1217	1345	1481	1625	1777	1937
17	613	689	773	865	965	1073	1189	1313	1445	1585	1733	1889	2053
18	685	765	853	949	1053	1165	1285	1413	1549	1693	1845	2005	2173
19	761	845	937	1037	1145	1261	1385	1517	1657	1805	1961	2125	2297
20	841	929	1025	1129	1241	1361	1489	1625	1769	1921	2081	2249	2425
21	925	1017	1117	1225	1341	1465	1597	1737	1885	2041	2205	2377	2557
22	1013	1109	1213	1325	1445	1573	1709	1853	2005	2165	2333	2509	2693
23	1105	1203	1313	1429	1553	1685	1825	1973	2129	2293	2465	2645	2833
24	1201	1305	1417	1537	1665	1801	1945	2097	2257	2425	2601	2785	2977
25	1301	1409	1525	1649	1781	1927	2069	2225	2389	2561	2741	2929	3125
26	1405	1517	1637	1765	1904	2045	2197	2357	2525	2701	2885	3077	3277
27	1513	1629	1753	1885	2025	2173	2329	2493	2665	2845	3033	3229	3433
28	1625	1745	1873	2009	2153	2305	2465	2633	2809	2993	3185	3385	3593

MISCELLANEOUS.

62. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

A tube of uniform cross section, small compared with its length, is bent into the form of a cycloid, its open ends lying at the cusps, and this cycloid is placed with its axis vertical and its vertex downwards. Equal quantities of fluids of specific gravity σ_1 and σ_2 are poured in at the two cusps, the quantity of each being such as would fill a length of the tube equal to its axis a. If the fluids do not mix, find the distance x_1, x_2 of the upper levels of the fluids from the vertex measured along the cycloidal arc. [From *Procter's Geometry of the Cycloid.*]

Solution by the PROPOSER.

Let P be the atmospheric pressure. Then the pressure at the first surface of separation is $P+g\sigma_1[x_1^2-(x_1-a)^2]/8a$.

Similarly at the second or free surface, $P+g\{\sigma_1[x_1^2-(x_1-a)^2]+\sigma_2[(x_1-a)^2-(x_1-2a)^2]\}/8a$.

$\therefore\ P=P+g\{\sigma_1[x_1^2-(x_1-a)^2]+\sigma_2[(x_1-a)^2-(x_1-2a)^2]\}/8a$.

$\therefore\ 2x_1(\sigma_1+\sigma_2)=a(\sigma_1+3\sigma_2)$.

Similarly $P=P+g\{\sigma_2[x_2^2-(x_2-a)^2]+\sigma_1[(x_2-a)^2-(x_2-2a)^2]\}/8a$.

$2x_2(\sigma_1+\sigma_2)=a(3\sigma_1+\sigma_2)$.

63. Proposed by F. M. SHIELDS, County Surveyor, Coopwood, Miss.

Of three chronometers, A, B, and C, A keeps true time; B gains 5 minutes and $15\frac{3}{4}$ seconds a day by true time, and C loses 7 minutes and $15\frac{3}{7}$ seconds a day by true time. The hands of all three watches are set at 12 noon on a certain day. What is the time by the true watch, A, on the *fifth* day after the time when the hands of the fast watch, B, point to 12, and what is the time by the true watch, A, on the *tenth* day after the time when the hands of the slow watch, C, point to 12?

I. Solution by J. H. DRUMMOND, LL. D., Portland, Me.

B goes $1445\frac{21}{80}$ minutes while A goes 1440; then $1445\frac{21}{80}:1440::5:$ 4d. 23h. $33\frac{90567}{115621}$m., and the answer is $33\frac{90567}{115621}$ minutes past eleven.

C goes $1432\frac{26}{35}$ while A goes 1440; hence $1432\frac{26}{35}:1440::10:$ 10d. 1h. $12\frac{23544}{25073}$m., and the answer is $12\frac{23544}{25073}$ minutes past one.

II. Solution by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.

While the watch A goes 24 hours; or 1440 minutes, the watch B goes 1445 minutes, $15\frac{3}{4}$ seconds, or $1445\frac{21}{80}$ minutes. To find how much the watch A is behind B at the end of 24 hours *by* B, we find 1 minute of B equal to $1445\div1445\frac{21}{80}$ of A's, and 1440 minutes of B's $=1440\div1445\frac{21}{80}=1434\frac{87486}{115621}$, *i. e.* when the hands of the watch B point to 12 at the end of the first 24 hours *by* B, the watch A lacks $5\frac{28135}{115621}$ minutes. On the *fifth* day after time when the hands of the watch B point to 12, the watch A lacks $5\frac{28135}{115621}\times5=26\frac{25054}{115621}$ minutes to 12.

At the end of the first 24 hours *by* C the watch A is $7\frac{7369}{25073}$ past 12. On the tenth day after time when the hands of the slow watch C point to 12, the time by the true watch A is $7\frac{7369}{25073}\times10=72\frac{23544}{25073}$ minutes, or 1 hour, $12\frac{23544}{25073}$ minutes past 12.

III. Solution by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

24 hours, 5 minutes and $15\frac{3}{4}$ seconds on chronometer B equals 24 hours on chronometer A. Hence 5 days on B equals 4 days, 23 hours, 33 minutes and 46.9 seconds on A. Therefore the time by the true watch A is 11 hours, 33 minutes, and 46.9 seconds A. M. on the fifth day.

$1432\frac{26}{35}$ minutes on chronometer C equals 1440 minutes on chronometer A.

Hence ten days on chronometer C equals 10 days, 1 hour, 12 minutes and 55 seconds on chronometer A. Therefore the time by the true watch A is 12 minutes, 55 seconds past 1 P. M. on the tenth day.

Also solved by *CHARLES C. CROSS*, and *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ALGEBRA.

90. **Proposed by J. MARCUS BOORMAN, Consultative Mechanician and Counselor at Law, Woodmere, Long Island, N. Y.**

Fully solve $x^2+y^2+m(x+y)=m^2$ ..... (I) ; $x^2+y^2+xy=m^2$ ...... (II) ; give process, roots, and corollary. [Solved *in part*, *Cirode's Algebra*, *page* 202.]

91. **Proposed by NELSON S. RORAY, Professor of Mathematics, South Jersey Institute, Bridgeton, N. J.**

Solve the following without making use of the determinant notation and prove that the results obtained are the roots.

$$10x-2y+4z=5,$$
$$3x+5y-3z=7,$$
$$x+3y-2z=2.$$

92. **Proposed by W. F. BRADBURY, A. M., Head Master Latin School, Cambridge, Mass.**

Find the sum to n terms of $1+3^3+5^3+$ ...... [From *Charles Smith's Elementary Algebra*, page 403.]

*** Solutions of these problems should be sent to J. M. Colaw not later than December 10.

GEOMETRY.

105. **Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

Sind A, B, C, D vier harmonische Punkte und beschreibt man über dem Durchmesser AC einen Kreis, von welchem S ein beliebiger Punkt ist, so wird derjenige Kreisbogen, welcher innerhalb des Winkels BSD liegt, entweder von A order von C Strecke PB halbirt. [*Reye's Geometrie der Lage*, page 191.]

106. **Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.**

Upon the sides of any triangle ABC let the equilateral triangles ABD, BCE, and CAF be described, and let their exterior sides produced intersect, BE and AF in K, DB and FC in L, and DA and EC in M. Prove DK, EL, FM, parallel.

107. **Proposed by T. W. PALMER, A. M., Professor of Mathematics, University of Alabama.**

Construct a triangle, given base, vertical angle and radius of inscribed circle.

*** Solutions of these problems should be sent to B. F. Finkel not later than December 10.

CALCULUS.

81. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Graduate Fellow in Mathematics in Vanderbilt University, Carthage, Texas.

Solve: $y^2(d^2y/dx^2)+a(dy/dx)^2=bx$.

82. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.

A pole 60 feet high stands vertically in a river 20 feet deep. How many feet above the surface of the water must it break so that the top bending down would touch the bottom and the distance on the surface of water between the points where the parts of the pole enter the water would be maximum?

*** Solutions of these problems should be sent to J. M. Colaw not later than December 10.

MECHANICS.

75. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A particle, P, is held in a bent tube by two forces directed towards two fixed points, H and S. Show that the equation to the form of tube is $PS.PH=k^2$, if the forces are μ/PS and μ/PH.

76. Proposed by J. F. LAWRENCE, Classical Sophomore, Drury College, Springfield, Mo.

An inclined plane of mass M is capable of moving freely on a smooth horizontal plane. A perfectly rough sphere of mass m is placed on its inclined face and rolls down under the action of gravity. If x' be the horizontal space advanced by the inclined plane, x the part of the plane rolled over by the sphere, prove that $(M+m)x'=mx\cos\alpha$, $\frac{7}{5}x-x'\cos\alpha=\frac{1}{2}gt^2\sin\alpha$, where α is the inclination of the plane. [From *Routh's Elementary Rigid Dynamics*, page 126.]

*** Solutions of these problems should be sent to B. F. Finkel not later than December 10.

DIOPHANTINE ANALYSIS.

74. Proposed by O. W. ANTHONY, M. Sc., Instructor in Mathematics, Boys' High School, New York City.

Solve
$$x^2+y^2=a^2,$$
$$z^2+w^2=b^2,$$
$$y^2+w^2=d^2,$$
i. e. obtain values of x, y, z, w, which will make the second members perfect squares.

75. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

(1). In how many ways can the consecutive integers 1 to 16 be arranged as a Magic Square?

(2). Arrange the consecutive integers 1 to n^2 as a Magic Square, where n is odd. Apply when $n=9$.

*** Solutions of these problems should be sent to J. M. Colaw not later than December 10.

AVERAGE AND PROBABILITY.

71. **Proposed by B. F. FINKEL, A. M., M. Sc.,Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

Find the average volume removed by boring an inch auger-hole through a cube whose edge is e, the auger to pass through two opposite faces of the cube.

72. **Proposed by B. F. FINKEL, A. M.,M. Sc.,Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

A rod is broken at random into four pieces; find the chance that no one of the pieces is greater than the sum of the other three. [From *C. Smith's Treatise on Algebra*, page 528.]

*** Solutions of these problems should be sent to B. F. Finkel not later than December 10.

MISCELLANEOUS.

68. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio State University, Athens, Ohio.**

Find the locus of the vertex of the cone enveloping the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ so that the plane of contact will constantly touch $x^2+y^2+z^2=r^2$.

69. **Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy, Pacific College, Santa Rosa, P. O., Sebastopol, Cal.**

Find the locus of a point equi-distant from the circumferences of two fixed circles.

*** Solutions of these problems should be sent to J. M. Colaw not later tnan December 10.

BOOKS.

Advanced Mechanics. Vol. I. *Dynamics.* The Organized Science Series. By William Briggs, M. A., F. E. S., F. R. A. S., and G. H. Bryan, Sc. D., F. R. S. 327 pages. Price, $1.00. 1898. London: W. B. Clive. New York: Hinds & Noble.

The notion of relative velocities is introduced at an early stage in the discussion of the laws of composition and resolution. The chapters on motion down inclined planes, chords of quickest descent, the parabolic path of a projectile, circular and harmonic motion, small oscillations of a pendulum, impact of elastic spheres, and the elements of rigid dynamics are highly satisfactory. There are numerous exercises. The plan of the book is good, and in the execution there is much to commend.

We have also received *Elementary Mechanics*, by the same authors, in the University Tutorial Series, which gives special prominence to fundamental principles, and is an especially good book of its grade. J. M. C.

Coördinate Geometry. The Right Line and Circle. By Briggs and Bryan. University Tutorial Series. Third Edition. 220 pages. Price, 80 cents.

In this elementary treatise the editors have tried "to realize the position of the average learner, and have constantly borne in mind the needs of the private student." The result is altogether satisfactory. J. M. C.

Mensuration and Spherical Geometry. By William Briggs, M. A., F. C. S., F. R. A. S., and T. W. Edmondson, B. A. Second Edition. $1.00. London: W. B. Clive. New York: Hinds & Noble.

The subject is not presented in the form of a mere collection of rules and examples, but in every case a proof of the formula to be used has been given. There are no material alterations in the revision, but some errors have been eliminated and some inaccuracies corrected. The work as now presented is well arranged and fairly comprehensive in treatment. It will be found a serviceable text. J. M. C.

Differential and Integral Calculus. By James M. Taylor, A. M., LL. D., Professor of Mathematics, Colgate University. Revised Edition. 1898. 8vo., cloth. xiii+269 pages. Price, $2.15. Boston: Ginn & Company.

In this revision of Professor Taylor's popular text the attempt has been made to present in their unity the three common methods in the Calculus. In his preface, the author says that "the concept of Rates is essential to a statement of the problems of the Calculus; the principles of Limits make possible general solutions of these problems, and the laws of Infinitesimals greatly abridge these solutions." The Method of Rates is so generalized and simplified that it does not involve "the foreign element of time," and affords simple and brief proofs of first principles. By proving $lt(\triangle y/\triangle x)=dy/dx$, the problem of rates is reducedto one of limits or infinitesimals. The Theory of Infinitesimals is viewed simply as that part of the theory of limits which deals with *variables having zero as their common limit.* Emphasis is also laid upon the distinction between infinitesimals and zero and that between infinites and $a/0$. A chapter on differential equations is an important addition to the text. The table of integrals is useful. Throughout the work are found numerous practical problems from mechanics and other branches of applied mathematics. One of the most distinctive features of this text (though it may be a source of objection to some) is the facility with which it may be used by teachers who may prefer either one of the three methods commonly used in the Calculus, as those who wish to pursue the study by the method of Limits or Infinitesimals alone can omit the few demonstrations which involve rates and take in their stead those by limits or infinitesimals. We bespeak for the book a hearty welcome from teachers. J. M. C.

Differential and Integral Calculus, for Technical Schools and Colleges. By P. A. Lambert, M. A., Assistant Professor of Mathematics, Lehigh University. 8vo. Cloth. x+246 pages. Price, $1.50. New York: The Macmillan Co.

This work is designed for students who have a working knowledge of Elementary Geometry, Algebra, Trigonometry and Analytical Geometry. The author states that the object of the book is: (1) By a logical presentation of principles to inspire confidence in the methods of the infinitesimal analysis; (2) By numerous problems to aid in acquiring facility in applying these methods; (3) By applications in Physics and Engineering, and other branches of Mathematics, to show the practical value of the Calculus.

Proposing problems belonging to Engineering is a good idea, and it would be well in making up lists of exercises to go in future works on the Calculus, to insert more problems in Economics and Engineering. B. F. F.

Text-Book of Algebra with Exercises for Secondary Schools and Colleges. By George Egbert Fisher, M. A., Ph. D., and Isaac J. Schwatt, Ph. D., Assistant Professors of Mathematics in the University of Pennsylvania. Part I. 8vo. Cloth. Price, $1.25. xiii+683 pages. Published by the authors.

This work begins with the very elements of algebra, establishing by rigorous demonstrations, and making clear by numerous illustrations, the fundamental Laws and Theorems, through the Fundamental Operations, Fractions, Equations of one, two, etc., unknown

quantities, Evolution, Inequalites, Quadratic Equations, Proportion and Variation, Progressions, and ends with a discussion of the Binominal Theorem for Positive Integral Exponents. Some of the strong points in favor of this work are: The solutions of equation based upon equivalent equations and equivalent systems of equations; the discussion of general problems and the interpretation of positive, negative, zero, indeterminate, and infinite solutions of problems; the logical discussion of irrational numbers; and the full treatment of inequalities. An unusually large number of problems and exercises have been added, in order that the teacher from year to year, may have variety with different classes. The work is one of unusual merit and is worthy the patronage of teachers desiring to put a good book into the hands of their pupils. The print is large and clear, the printed page is not overcrowded, and the binding is good. B. F. F.

Introduction to the Theory of Analytic Functions. By J. Harkness, M. A. (Cambridge), Professor of Mathematics, Bryn Mawr College, Pa., and F. Morley, Sc. D. (Cambridge), Professor of Mathematics, Haverford College, Pa. Large 8vo. Cloth. xv + 336 pages. Price, $3.00. New York: The Macmillan Co.

This work, as its name indicates, is an introduction to the Theory of Functions. Until the translation of Durege's Elements of the Theory of Functions by Drs. Fisher and Schwatt, in 1896, there was no work in English giving a consecutive and elementary account of the fundamental concepts and processes employed in the Theory of Functions. The translation referred to above has been very helpful in aiding the student in making some headway in the larger works in English, viz., Forsyth's Treatise, and Harkness and Morley's Treatise. To still further aid in introducing the student to the advanced works, the work before us is intended. The work begins with a treatment of the Ordinal Number System. This discussion while very elementary, is of the highest importance, and might well be incorporated in some works on arithmetic, for it is true that few arithmetics at present bear evidence of the great progress and improvement made in mathematics. Chapter II treats of the Geometric Representation of Complex Number; Chapter III, of the Bilinear Transformation; Chapter IV, of Geometric Theory of the Logarithm and the Exponential; Chapter V, of the Bilinear Transformation of the Plane into Itself; Chapter VI, of Limits and Continuity; Chapter VII, of Rational Algebraic Functions; Chapter VIII, of Convergence of Infinite Series; Chapter IX, of Uniform Convergence of Real Series; Chapter X, of Power Series; Chapter XI, of Operations with Power Series; Chapter XII, of Continuation of Power Series; Chapter XIII, of Analytic Theory of the Exponential and Logarithm; Chapter XIV, of Singular Points of Analytic Functions; Chapter XV, of Weirstrass's Factor-Theorem; Chapter XVI, of Integration; Chapter XVII, of Laurant's Theorem and the θ-Functions; Chapter XVIII, of Functions Arising from a Network; Chapter XIX, of Elliptic Functions; Chapter XX, of Simple Algebraic Functions on Reimann Surfaces; Chapter XXI, of Algebraic Functions; Chapter XXII, of Cauchy's Theory and the Potential. The work forms an excellent introduction to the authors' Treatise. B. F. F.

An Elementary Course in Analytical Geometry. By J. H. Tanner, Assistant Professor of Mathematics in Cornell University, and Joseph Allen, formerly Instructor in Mathematics in Cornell University, Tutor in the College of the City of New York. 8vo. Cloth. xx + 390 pages. Price, $2.00. New York, Cincinnati, and Chicago: The American Book Co.

This is the first of the Cornell Mathematical Series. In addition to the usual discussion of the Conic Section. there is 22 pages devoted to the discussion of Higher Plane Curves; 50 pages to Solid Analytical Geometry, and an Appendix of 10 pages in which is a short historical account of the invention of Analytical Geometry, discussions of the limiting cases of series, and the demonstration of the proposition due to Hamilton, Quételet, and others. The work is interspersed with brief historical notes adding much interest to the study of the subject. B. F. F.

ERRATA.

Page 106, line 20, for "2(3)" read $2(3)^2$; line 21, for "$2(a^4+b^4+c^2)$" read $2(a^4+b^4+c^4)$; line 29, last term of denominator, for "$3(a^2+b^2+c^2)^2$" read $3(a^2+b^2-c^2)^2$.

Page 107, bracket together lines 9, 10 and 11, and number (15); line 22, for "$a^4c^2+2a^2c^4-c^6$" read $a^4c^2-2a^2c^4+c^6$.

Page 118, line 2, for "$\cos n\alpha$)" read $\cos n^2\alpha$).

Page 139, line 3 from bottom, for "$P_2 a^{n-4}$" read $P_2 a_2{}^{n-4}$.

Page 140, line 2, for "P_{n-2}" read $\pm P_{n-2}$; line 6, in second term for "$a^r a_1{}^{n-2}$" read $a_2{}^r a_1{}^{n-2}$.

Page 151, line 28, where "18" occurs read 78.

Page 155, line 6 from bottom, in numerator of (A), for "$(n-2)$" read $(n-1)$.

Page 157, line 8, for "$ED : DB$" read $EB : OB$.

Page 159, problems 62 and 63 should be 64 and 65.

Page 173, line 30, for "$-2[S$" read $\pm 2[8$.

Page 174, line 3, for "946268" read 04268; line 5, for "8" read -8; line 6, for "0.512372" read ∓ 15.487627; line 9, for "-0.064568," etc., read 0.064568, etc.; line 10, for "300343" read 800343; line 18, for "$\frac{1}{4}f\sqrt{[1(a^2-b)]}$" read $\frac{1}{4}f\sqrt{[1/(a^2-b)]}$; lines 19 and 20, for "$(\frac{1}{4}+)$" read $(1/4+)$.

Page 175, line 2, of problem 92, for "$AB.BC : DC.AD=BD : AC$" read $AB.BC+AD.CD : AB.AD+BC.CD :: BD : AC$.

Page 177, line 5, for "EBC" read FBC; line 12, insert sign $=$ before $\frac{1}{4}(2880-y^2)$; line 21, "353.3604" read 353.8604; supply F in figure.

Page 180, line 13, insert comma after $(2n^2+4n+1)^2$; line 29, for "$\sqrt{[(q^2+4q^2+4q+1)}$" read $\sqrt{[(q^3+4q^2+4q+1)}$.

Page 181, line 5, for "$-3m^3a^2y$" read $-3m^2u^2y$; line 17, problem should be 64.

Page 182, line 2 from bottom, read $(y/r)^{\frac{2}{3}}=a^2/(1+a^2)$.

Page 183, line 2, for (3) read $a/(1-a^2)=\{(y/r)^{\frac{1}{3}}\sqrt{[1-(y/r)^{\frac{2}{3}}]}\}/[1-2(y/r)^{\frac{2}{3}}]$; line 19, for "$\alpha\chi$" read α/χ; line 3 from bottom, for "$y^3=\frac{1}{2}y+.094119=0$" read $y^3-\frac{1}{2}y+.094118=0$.

Page 186, line 4, insert *of* after "value"; line 6, for "α" read ∞; in problem 79, where ε occurs insert e.

Page 187, problems 64 and 65 should be 66 and 67.

Page 201, line 12 from bottom, "10 chains" should be 10 rods.

Page 203, line 21, for "$1/m^2.(m^2)!$" read $3/m^2.(m^2)!$; line 22, insert $=$ before $\pi/6$.

Page 204, in lines 1, 2, 3, 4, and 5, insert the sign $=$ before the terms containing π^2 in the numerators.

Page 205, last line of Solution II., for "$a(a+b)$" read $x^2=-a(a+b)$.

Page 206, line 1, for "$\angle BCD$" read $\angle BDC$; for denominator of $\tan^{-1}$ read $c^2+a^2+b^2$.

Page 214, last line, for "36^2" read 35^2.

Page 215, line 6, for "p^2" read p.

In advertisement of Open Court Publishing Co., price of *Monist* should be $2.00.

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AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. NOVEMBER, 1898. No. 11.

POINT INVARIANTS FOR THE FINITE CONTINUOUS GROUPS OF THE PLANE.

By DAVID A. ROTHROCK, Indiana University, Bloomington, Indiana.

In an article entitled "Theorie der Transformationsgruppen," Mathematische Annalen, Bd. XVI., Lie has classified and reduced to canonical forms all the *finite continuous* groups of the plane. We may very appropriately call these the *Lie groups*. In the present paper it is proposed to determine the functions of the coördinates of n points which remain invariant by these groups. In some cases the computations are quite complex; only the methods of calculation are given.

An infinitesimal point transformation gives to x and y the increments

$$\delta x=\xi(x, y)\delta t, \quad \delta y=\eta(x, y)\delta t,$$

respectively, where δt is independent of x and y. Such an infinitesimal transformation is denoted in the Lie notation by

$$Xf\equiv\xi(x, y)\frac{\partial f}{\partial x}+\eta(x, y)\frac{\partial f}{\partial y}. \tag{1}$$

The variation of any function $\varphi(x, y)$ by this transformation is given by $X\varphi(x, y)\delta t$. Hence, if $\varphi(x, y)$ is to remain invariant by the transformation Xf, its variation must be zero, or

$$X\varphi \equiv \xi(x,\ y)\frac{\partial \varphi}{\partial x}+\eta(x,\ y)\frac{\partial \varphi}{\partial y}=0.$$

The function $\varphi(x,\ y)$, invariant by the infinitesimal transformation Xf, is determined as a solution of the linear partial differential equation $Xf=0$.

The infinitesimal transformation Xf may be *extended* to include the increments of any number of points, x_i, y_i, $i=1,\ 2 \ldots\ldots n$. We may write this extended transformation as

$$Wf \equiv \sum_1^n iX^{(i)}f \equiv \sum_1^n i\xi(x_i,\ y_i)\frac{\partial f}{\partial x_i}+\sum_1^n i\eta(x_i,\ y_i)\frac{\partial f}{\partial y_i}. \tag{2}$$

The functions of the coördinates of n points invariant by Wf will be the $2n-1$ independent solutions of $Wf=0$. n of these solutions may be selected in the form $\varphi(x_i,\ y_i)$, where $\varphi(x,\ y)$ is a solution of $Xf=0$; the remaining $n-1$ solutions will in general differ from $\varphi(x,\ y)$ in form.*

Infinitesimal transformations are called *independent* when no one can be expressed as a linear function of the others with constant coefficients. r such independent infinitesimal transformations

$$X_kf \equiv \xi_k(x,\ y)\frac{\partial f}{\partial x}+\eta_k(x,\ y)\frac{\partial f}{\partial y},\ (k=1\ \ldots.r),$$

form a group when

$$X_i(X_kf)-X_k(X_if) \equiv \sum_1^r sc_{iks}X_sf,\ (c_{iks}=\text{constants}). \tag{3}$$

The transformations of this *r-parameter* group X_kf may be extended after the manner of (2) above, giving us W_kf which determine the increments of a function $f(x_1,\ y_1;\ x_2,\ y_2;\ \ldots\ldots x_n,\ y_n)$. On account of relation (3),

$$W_1f=0,\ \ W_2f=0,\ \ldots\ldots W_rf=0$$

are known to form a *complete* system of linear partial differential equations in $2n$ variables $x_1y_1,\ x_2y_2,\ \ldots\ldots x_ny_n$, with at least $2n-r$ independent solutions. If the r-rowed determinants of the coefficients vanish, more than $2n-r$ independent solutions will exist. These solutions are the invariants of the coördinates of n points by the r-parameter group of X_kf.

According to the theory here outlined we shall determine the *point-invariants* of the finite continuous groups as classified by Lie in the Mathematische Annalen, Vol. XVI.

*Lie: Theorie der Transformationsgruppen, Bd. I., §59.

§ 1.
Invariants of the Primitive Groups.

The *primitive* groups of the plane leave *no* family of ∞^1 curves invariant, and may be reduced by a proper choice of variables to some one of the canonical forms known as (1) special linear, (2) general linear, (3) general projective.

1. *The special linear group*

$$\overline{p,\ q,\ xq,\ xp-yq,\ yp}\,\Big|\,.$$

Here $p=\frac{\partial f}{\partial x}$, $q=\frac{\partial f}{\partial y}$. The increments of $f(x_i, y_i)$ by the members of this group *extended* are given by

$$W_1f\equiv\sum_1^n i\frac{\partial f}{\partial x_i},\quad W_2f\equiv\sum_1^n i\frac{\partial f}{\partial y_i},\quad W_3f\equiv\sum_1^n ix_i\frac{\partial f}{\partial y_i},$$

$$W_4f\equiv\sum_1^n i\left(x_i\frac{\partial f}{\partial x_i}-y_i\frac{\partial f}{\partial y_i}\right),\quad W_5f\equiv\sum_1^n iy_i\frac{\partial f}{\partial x_i}.$$

The invariant functions sought will be the $2n-5$ independent solutions of the complete system

$$W_1f=0,\quad W_2f=0,\quad W_3f=0,\quad W_4f=0,\quad W_5f=0. \tag{s}$$

The first two equations show the solutions to be functions of

$$\varphi_j=x_1-x_j,\quad \psi_j=y_1-y_j,\quad (j=2\ldots\ldots n).$$

The remaining equations then take the form

$$\sum_2^n j\varphi_j\frac{\partial f}{\partial \psi_j}=0,\quad \sum_2^n j\left(\varphi_j\frac{\partial f}{\partial \varphi_j}-\psi_j\frac{\partial f}{\partial \psi_j}\right)=0,\quad \sum_2^n j\psi_j\frac{\partial f}{\partial \varphi_j}=0.$$

The second of these equations has solutions

$$\Phi_j=\varphi_j\psi_j,\quad \Psi_k=\varphi_2\psi_k,\quad (k=3\ldots\ldots n).$$

With these functions as new variables, the first and third equations become

$$\frac{\partial f}{\partial \phi_2}+\sum_3^n k\frac{\phi_k}{\Psi_k}\left\{\frac{\phi_k}{\Psi_k}\frac{\partial f}{\partial \phi_k}+\frac{\partial f}{\partial \psi_k}\right\}=0, \tag{1}$$

$$\frac{\phi_2}{\Psi_3}\frac{\partial f}{\partial \phi_2}+\frac{1}{\phi_2\psi_3}\sum_3^n k\psi_k^2\frac{\partial f}{\partial \varphi_k}+\frac{1}{\psi_3}\sum_3^n k\psi_k\frac{\partial f}{\partial \psi_k}=0. \tag{2}$$

The solutions of (1) are seen to be

$$\sigma_k = \frac{\psi_k}{\varphi_k}, \quad \rho_k = \varphi_2 - \frac{\psi_k^2}{\varphi_k},$$

while (2) takes the form

$$\sum_3^n k\left(\sigma_k \rho_k \frac{\partial f}{\partial \sigma_k} + \rho_k^2 \frac{\partial f}{\partial \rho_k}\right) = 0,$$

with solutions

$$\varDelta_k = -\frac{\rho_k}{\sigma_k}, \quad I_e = \frac{1}{\rho_e} - \frac{1}{\rho_s}, \quad (k=3 \ldots\ldots n, \; l=4 \ldots\ldots n).$$

Since any functions of $\varDelta$ and I will be solutions of our complete system (s), we may choose

$$\varDelta_k = -\frac{\rho_k}{\rho_2} = |\; 1 \;\; 2 \;\; k \;|, \quad D_l = I_l . \varDelta_3 . \varDelta_l = |\; 1 \;\; 3 \;\; l \;|, \quad \text{where } |\; i \; j \; k \;| = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix},$$

as solutions of (s), and, therefore, as the $2n-5$ invariants of the coördinates of n points.

The forms of $\varDelta$ and D show that *the special linear group leaves invariant all areas.*

2. *The general linear group*

$$\boxed{p, \; q, \; xq, \; xp-yq, \; yp, \; xp+yq}$$

This group furnishes a complete system of six linear partial differential equations, the first five equations of the system being identical with those of the preceding section. We need only to determine the functions of $\varDelta_k$ and D_l which satisfy

$$\sum_1^n i \left(x_i \frac{\partial f}{\partial x_i} + y_i \frac{\partial f}{\partial y_i}\right) = 0.$$

This equation requires that x, y enter in the final solutions with the degree zero. Hence, we may write at once the invariants:

$$I_l = \frac{\varDelta_l}{\varDelta_3} = |\; 1 \;\; 2 \;\; l \;| : |\; 1 \;\; 2 \;\; 3 \;|$$

$$J_l = \frac{D_l}{\varDelta_3} = |\; 1 \;\; 3 \;\; l \;| : |\; 1 \;\; 2 \;\; 3 \;| \qquad (l=4, \;\ldots\; n).$$

Hence, *by the general linear group the ratio of areas remain constant.*

3. *The general projective group*

$$\boxed{p, \; q, \; xq, \; xp-yq, \; yp, \; xp+yq, \; x^2p+xyq, \; xyp+y^2q}.$$

The members of this group extended and equated to zero give a complete system of eight partial differential equations, the first six of which are identical with those of the general linear group, and therefore have solutions I, J defined above. The last two equations,

$$\sum_1^n ix_i(x_i\frac{\partial f}{\partial x_i}+y_i\frac{\partial f}{\partial y_i})=\sum_1^n iy_i(x_i\frac{\partial f}{\partial x_i}+y_i\frac{\partial f}{\partial y_i})=0,$$

expressed in terms of I and J, become somewhat complex. For brevity we shall write these symbolically as

$$Uf=0, \quad Vf=0,$$

where $Uf=0$ has the solutions

$$I_4, \quad \Phi_m=\frac{I_m J_4}{J_m}, \quad \Psi_m=\frac{\phi_m+I_m(I_4-\phi_m-1)}{I_m(J_4-I_4+1)}, \quad (m=5.\,.\,.\,.\,.n).$$

With I_4, ϕ_m and ψ_m as new variables, the equation $Vf=0$ takes the simple form

$$I_4\frac{\partial f}{\partial I_4}+\sum_5^n m\varphi_m\frac{\partial f}{\partial \varphi_m}=0,$$

whose solutions are clearly

$$\theta_m=\frac{\varphi_m}{I_4}, \quad \Psi_m, \quad (m=5.\,.\,.\,.\,.\,.n).$$

Selecting as invariants θ_m and $H_m=\dfrac{1+\Psi_m}{\theta_m}$, and restoring the variables x_i, y_i we have

$$\theta_m=\frac{|\,1\;2\;m\,|\,.\,|\,1\;3\;4\,|}{|\,1\;2\;4\,|\,.\,|\,1\;3\;m\,|}, \quad H_m=\frac{|\,1\;2\;4\,|\,.\,|\,2\;3\;m\,|}{|\,1\;2\;m\,|\,.\,|\,2\;3\;4\,|}.$$

The forms of θ and H show *that the general projective group leaves invariant the cross-ratios of five points.* Five points have two independent invariants, θ, H; four points have no invariant unless they be collinear, in which case the invariant is the cross-ratio of the four points.

§ 2.

Invariants of such Imprimitive Groups as Leave Unchanged One Family of ∞^1 Curves.

The remaining finite continuous groups of the plane are known as *imprimitive*, and are classified according as they leave invariant *one*, *two*, or an *infinite number* of families of ∞^1 curves.* The groups of the first category have been

*Lie: Math. Annalen, Bd. XVI.

reduced by Lie to such canonical forms that the family of curves x=constant is transformed into itself.

4. $$\boxed{\begin{array}{c} X_1q,\ X_2q,\ X_3q,\ \ldots\ldots X_rq \\ r>1 \end{array}}.$$

Here X_k is a function of x alone. This group leaves the curves of the family $x=a$ singly invariant.

The complete system

$$W_kf \equiv \sum_1^n iX_k(x_i)\frac{\partial f}{\partial y_i}=0,\quad (k=1\ \ldots.\ r),$$

corresponding to this group, has as solutions $x_1, x_2, \ldots.. x_n$ and $n-r$ other independent functions $\varDelta_s$, $(s=1\ldots..\ n-r)$, which we shall define as the $n-r$ determinants of the matrix

$$\left\|\begin{array}{llll} y_1 & y_2 & y_3 & \cdots y_{r+s} \cdots\cdots y_n \\ X_1(x_1) & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ X_2(x_1) & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ X_r(x_1) & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{array}\right\|,\qquad (M_r)$$

$$(n \gtreqless r+1)$$

formed by filling the $(r+1)$th column successively by the $(r+1)$, the $(r+2)\ldots.$, and the nth.

Our invariants are x_i and $\varDelta_s$, $(i=1\ldots\ldots n,\ s=1\ldots\ldots n-r)$.

5. $$\boxed{\begin{array}{c} X_1q,\ \ X_2q\ldots\ldots X_{r-1}q,\ \ yq \\ r>2 \end{array}}.$$

This group furnishes the complete system

$$W_kf \equiv \sum_1^n iX_k(x_i)\frac{\partial f}{\partial y_i}=0,\quad Yf \equiv \sum_1^n iy_i\frac{\partial f}{\partial y_i}=0,\quad (k=1\ldots\ldots r-1).$$

The solutions of $W_kf=0$ are clearly $x_1, x_2, \ldots\ldots x_n$ and the determinants $\varDelta_s$, $(s=0,\ 1,\ \ldots\ldots n-r)$, of the matrix (M_{r-1}) defined above. $Yf=0$ requires the ratios of y_i to appear. We may then write as invariants of the group

$$x_1,\ x_2,\ \ldots\ldots x_n \text{ and } \rho_t=\frac{\varDelta_t}{\varDelta_0},\quad (t=1\ldots\ldots n-r).$$

6. $$\boxed{\begin{array}{c} \varepsilon^{\alpha_k x}q,\ \ x\varepsilon^{\alpha_k x}q,\ \ x^2\varepsilon^{\alpha_k x}q,\ \ldots\ldots x^{\rho_k}\varepsilon^{\alpha_k x}q,\ p \\ k=1\ldots\ldots m,\ \ \sum_1^m k\rho_k+m=r-1,\ \ r>2 \end{array}}.$$

We have

$$W_k^{t_k}f\equiv\sum_1^n i(x_i)^{t_k}\,.\,\varepsilon^{\alpha_k x_i}\frac{\partial f}{\partial y_i}=0,\ \ (k=1,\ \ldots\ldots m,\ t_k=0,\ 1\ldots\ldots\rho_k).$$

$$Xf\equiv\sum_1^n i\frac{\partial f}{\partial x_i}=0.$$

The last equation requires the functions

$$\varphi_j=x_j-x_1\quad (j=2\ldots\ldots n),$$

to appear in the solutions of the system. On dividing the remaining equations, respectively, by $\varepsilon^{\alpha_t x_1}$, the exponents of ε all become functions of φ_j. The independent determinants Δ_s, $(s=0,\ 1\ldots\ldots n-r)$ of the matrix (M_{r-1}), formed as in 4, will be solutions.

The invariants are, therefore, φ_j and Δ_s.

7. $$\boxed{\begin{array}{c} \varepsilon^{\alpha_k x}q,\ \ x\varepsilon^{\alpha_k x}q,\ \ x^2\varepsilon^{\alpha_k x}q,\ \ldots\ldots\ x^{\rho_k}\varepsilon^{\alpha_k x}q,\ yq,\ p \\ k=1,\ \ldots\ldots\ m,\ \ \sum_1^n k\rho_\kappa+m=r-2,\ \ r>3 \end{array}}$$

The complete system given by this group is

$$W_k{}^{t_k}f\equiv\sum_1^n i(x_i)^{t_k}\,.\,\varepsilon^{\alpha_k x_i}\frac{\partial f}{\partial y_i}=0,\ \ (k=1\ldots\ldots m,\ t_k=0,\ 1,\ \ldots\ldots\rho_k).$$

$$Yf\equiv\sum_1^n iy_i\frac{\partial f}{\partial y_i}=0,\ \ Xf\equiv\sum_1^n i\frac{\partial f}{\partial x_i}=0.$$

If a matrix be constructed, as indicated in 4 and 6, from the coefficients of the first $r-2$ equations, it will be observed that the independent determinants Δ_s, $(s=-1,\ 0,\ 1\ldots\ldots n-r)$, will be linear and homogeneous in y_i with coefficients composed of functions of $\varphi_j=x_j-x_1$. Δ_s will be solutions of all equations except $Yf=0$, which requires the ratios of y_i to appear. Hence, the invariants may be written

$$\varphi_j=x_j-x_1,\ \ \rho_t=\frac{\Delta_t}{\Delta_{-1}},\ \ (j=2\ldots\ldots n,\ \ t=0\ldots\ldots\ n-r).$$

8. $$\boxed{\begin{array}{c} q, \quad xq, \quad x^2q, \ldots\ldots x^{r-3}q, \quad p, \quad xp+cyq \\ r>3 \end{array}}.$$

Here the complete system is

$$W_k f \equiv \sum_1^n i x_i^k \frac{\partial f}{\partial y_i} = 0, \quad (k=0,\ 1\ \ldots\ldots r-3)$$

$$Yf \equiv \sum_1^n i x_i \frac{\partial f}{\partial x_i} + c\sum_1^n i y_i \frac{\partial f}{\partial y_i} = 0, \quad Xf \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0.$$

The solutions

$$\psi_j = y_1 - y_j, \quad \varphi_j = x_1 - x_j, \quad (j=2\ldots\ldots n),$$

of the first and last of these equations, introduced in Yf, give

$$Y'f \equiv \sum_2^n j\,(\varphi_j \frac{\partial f}{\partial \varphi_j} + c\,\psi_j \frac{\partial f}{\partial \psi_j}) = 0,$$

whose solutions are

$$\sigma_k = \frac{\varphi_k}{\varphi_2}, \quad \rho_j = \frac{\psi_j}{(\varphi_j)^c}, \quad (k=3\ldots\ldots n, \ j=2\ldots\ldots n).$$

$W_k f$, expressed in σ, ρ, is

$$W_t'f \equiv \frac{\partial f}{\partial \rho_2} + \sum_3^n k\sigma_k^{t-c} \frac{\partial f}{\partial \rho_k} = 0, \quad (t=1, \ \ldots\ldots r-3).$$

The solutions of these $r-3$ equations are σ_k, and the determinants Δ_s, $(s=1, 2\ldots\ldots n-r+2)$, of the matrix

$$\left\|\begin{array}{ccccc} \rho_2 & \rho_3 & \rho_4 \ldots\ldots \rho_{r-3+s} \ldots\ldots & & \rho_n \\ 1 & \sigma_3^{1-c} & \sigma_4^{1-c} \ldots\ldots\ldots\ldots\ldots\ldots & & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & & \\ & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & & \\ 1 & \sigma_3^{r-3-c} & \sigma_4^{r-3-c} \ldots\ldots\ldots\ldots & & \sigma_n^{r-3-c} \end{array}\right\|.$$

Our invariants are

$$\sigma_k = \frac{x_1 - x_k}{x_1 - x_2} \text{ and } \Delta_s.$$

9. $$\boxed{\begin{array}{c} q, \quad xq, \quad x^2q, \ldots\ldots x^{r-3}q, \quad p, \quad xp+[(r-2)y+x^{r-2}]q \\ r>2 \end{array}}.$$

The solutions of the complex system,

$$W_k f \equiv \sum_1^n i\, x_i^k \frac{\partial f}{\partial y_i} = 0, \quad (k=0,\ 1 \ldots\ldots r-3),$$

$$Yf \equiv \sum_1^n i \{ x_i \frac{\partial f}{\partial x_i} + [\,(r-2) y_i + x_i^{r-2}] \frac{\partial f}{\partial y_i} \} = 0,$$

$$Xf \equiv \sum_1^n i \frac{\partial f}{d x_i} = 0,$$

may be obtained in a manner similar to 8. The solutions, ψ_j, φ_j, of the first and last equations, introduced in Yf as variables, give

$$Y'f \equiv \sum_2^n j\, \{ \varphi_j \frac{\partial f}{\partial \varphi_j} + [(r-2)\psi_j + \varphi_j^{r-2}] \frac{\partial f}{\partial \psi_j} \} = 0,$$

with solutions,

$$\sigma_k = \frac{\varphi_k}{\varphi_2}, \quad \rho_j = \log \varphi_j - \frac{\psi_j}{\varphi_j^{r-2}}, \quad (k=3 \ldots\ldots n,\ j=2 \ldots\ldots n).$$

Introducing σ, ρ as new variables in Wf, we have

$$W_t' f \equiv \frac{\partial f}{\partial \rho_2} + \sum_3^n k \sigma_k^{t+2-r} \cdot \frac{\partial f}{\partial \rho_k} = 0,$$

whose solutions are σ_k and J_s of the matrix constructed as in 8.

The invariants are, therefore,

$$\sigma_k = \frac{x_1 - x_k}{x_1 - x_2}, \quad J_s, \quad (k=3 \ldots\ldots n, \quad s=1 \ldots\ldots n-r+2).$$

10. $\boxed{q,\ xq,\ x^2 q,\ \ldots\ldots x^{r-4} q,\ yq,\ p,\ xp \quad r>3}$.

For this group

$$W_t f \equiv \sum_1^n i x_i^t \frac{\partial f}{\partial y_i} = 0, \quad (t=0,\ 1 \ldots\ldots r-4),$$

$$Yf \equiv \sum_1^n i y_i \frac{\partial f}{\partial y_i} = 0; \quad X_1 f \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0, \quad X_2 f \equiv \sum_1^n i x_i \frac{\partial f}{\partial x_i} = 0.$$

The last two equations show that the ratios of the differences of the x's, say

$$\sigma_k = \frac{x_1 - x_k}{x_1 - x_2}, \quad (k=3, \ldots\ldots n),$$

shall appear in the final solutions. The $n-r+3$ independent determinants Δ_s, $(s=0, 1 \ldots\ldots n-r+2)$, of the matrix

$$\left\| \begin{array}{lllll} y_1 & y_2 & y_3 \cdots\cdots & y_{r-2+s} \cdots\cdots & y_n \\ 1 & 1 & 1 \ldots\ldots & 1 \ldots\ldots\ldots\ldots & 1 \\ x_1 & x_2 & x_3 \ldots\ldots & \ldots\ldots\ldots\ldots & x_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^{r-4} & x_2^{r-4} \ldots & \ldots & \ldots x_{r-2+s}^{r-4} \ldots & x_n^{r-4} \end{array} \right\|$$

are solutions of the first $r-3$ equations. These determinants are at the same time homogeneous in y_i and $x_i - x_k$; their ratios will, therefore, satisfy the requirements of

$$\sigma_k = \frac{x_1 - x_k}{x_1 - x_2} \text{ and } Yf=0.$$

Hence, we may write our $2n-r$ invariants as

$$\sigma_k \text{ and } \rho_t = \frac{\Delta_t}{\Delta_0}, \quad (k=3 \ldots\ldots n, \ t=1 \quad \ldots . n-r+2).$$

11. $q, \quad xq, \quad x^2 q, \ \ldots\ldots x^{r-4} q, \quad p, \quad 2xp + (r-4)yq, \quad x^2 p + (r-4)xyq$
$r > 4$.

From this group we obtain the differential equations

$$W_t f \equiv \sum_1^n i x_i^t \frac{\partial f}{\partial y_i} = 0, \quad (t=0, \ 1 \ldots\ldots r-4),$$

$$X_1 f \equiv \sum_1^n i \left(2x_i \frac{\partial f}{\partial x_i} + (r-4)\, y_i \frac{\partial f}{\partial y_i}\right) = 0,$$

$$X_2 f \equiv \sum_1^n i \left(x_i^2 \frac{\partial f}{\partial x_i} + (r-4)\, x_i y_i \frac{\partial f}{\partial y_i}\right) = 0,$$

$$X_0 f \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0.$$

With the solutions ψ_j, φ_j, of the first and last of these equations, $X_2 f$ takes the form

$$X_2' f \equiv \sum_2^n j \left(\varphi_j^2 \frac{\partial f}{\partial \varphi_j} + (r-4)\, \varphi_j \psi_j \frac{\partial f}{\partial \psi_j}\right) = 0,$$

whose solutions may be selected,

$$u_k = \frac{1}{\varphi_2} - \frac{1}{\varphi_k}, \quad v_j = \frac{\psi_j}{\varphi_j{}^{r-4}}, \quad (k=3\ldots\ldots n,\ j=2\ldots\ldots n).$$

$X_1 f$ then becomes

$$X_1{}' f \equiv 2\sum_3^n k u_k \frac{\partial f}{\partial u_k} + (r-4)\sum_2^n j v_j \frac{\partial f}{\partial v_j} = 0,$$

with solutions

$$\sigma_l = \frac{u_l}{u_3}, \quad \rho_k = \frac{v_k}{u_k{}^{\frac{1}{2}(r-4)}}, \quad \rho_2 = \frac{v_2}{u_3{}^{\frac{1}{2}(r-4)}}, \quad (l=4\ \ldots\ldots n,\ k=3\ \ldots\ldots n).$$

The remaining equations Wf in terms of σ, ρ are

$$\frac{\partial f}{\partial \rho_2} + \frac{\partial f}{\partial \rho_3} + \sum_4^n l(\sigma_l)^{-\alpha}\frac{\partial f}{\partial \rho_l} = 0, \quad [\alpha = \tfrac{1}{2}(r-4)],$$

$$W_t{}' f \equiv \frac{\partial f}{\partial \rho_3} + \sum_4^n l(\sigma_l)^{t-\alpha}\frac{\partial f}{\partial \rho_l} = 0, \quad (t=1,\ \ldots\ldots r-5).$$

Constructing a matrix,

$$\left\|\begin{array}{lll} \rho_2 & \rho_3 & \rho_4\ \ldots\ldots\ldots\ \rho_{r-3+s}\ \ldots\ldots\ \rho_n \\ 1 & 1 & \sigma_4{}^{-\alpha}\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 1 & \sigma_4{}^{1-\alpha}\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 1 & \sigma_4{}^{r-5-\alpha}\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\|$$

our invariants may be written

$$\sigma_l = \frac{x_2 - x_l}{x_1 - x_l} : \frac{x_2 - x_3}{x_1 - x_3}, \text{ and } \Delta_s, \quad (l=4\ldots\ldots n,\ s=1\ldots\ldots n-r+3).$$

12. $\boxed{q,\ xq,\ x^2q,\ \ldots\ldots x^{r-5}q,\ yq,\ p,\ xp,\ x^2p+(r-5)xyq \quad r>5}$.

This group furnishes the system

$$W_t f \equiv \sum_1^n i x_i^t \frac{\partial f}{\partial y_i} = 0, \quad (t=0,\ 1\ldots\ldots r-5), \quad Yf \equiv \sum_1^n i y_i \frac{\partial f}{\partial y_i} = 0,$$

$$X_2 f \equiv \sum_1^n i \left(x_i^2 \frac{\partial f}{\partial x_i} + (r-5)\, x_i y_i \frac{\partial f}{\partial y_i}\right) = 0,$$

$$X_1 f \equiv \sum_1^n i x_i \frac{\partial f}{\partial x_i} = 0, \qquad X_0 f \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0.$$

The first and last two equations show the final solutions to be functions of

$$u_k = \frac{x_1 - x_k}{x_1 - x_2}, \quad \psi_j = y_1 - y_j, \quad (k=3 \ldots\ldots n,\ j=2. \ \ldots . n).$$

$X_2 f$ now becomes

$$X_2' f \equiv \sum_3^n k u_k (u_k - 1) \frac{\partial f}{d u_k} + (r-5) \sum_3^n k u_k \frac{\partial f}{\partial \psi_k} + (r-5)\psi_2 \frac{\partial f}{\partial \psi_2} = 0,$$

with solutions

$$\sigma_l = \frac{u_3(u_l - 1)}{u_l(u_3 - 1)}, \quad \rho_k = \frac{\psi_k}{(u_k - 1)^{r-5}}, \quad \rho_2 = \psi_2 \left(\frac{u_3}{u_3 - 1}\right)^{r-5}, \ (l=4 \ \ldots\ldots n,\ k=3 \ \ldots\ldots n).$$

The remaining equations expressed in the new variables take the form

$$\left.\begin{aligned} &\frac{\partial f}{\partial \rho_2} + \frac{\partial f}{\partial \rho_3} + \sum_4^n l\, (\sigma_l)^{5-r} \frac{\partial f}{\partial \rho_l} = 0, \\ &W_t' f \equiv \frac{\partial f}{\partial \rho_3} + \sum_4^n l (\sigma_l)^{-t} \frac{\partial f}{\partial \rho_l} = 0, \ (t=1, \ \ldots\ldots r-4). \end{aligned}\right\} (A).$$

$$Y' f \equiv \sum_2^n j\, \rho_j \frac{\partial f}{\partial \rho_j} = 0.$$

The independent determinants Δ_s, $(s=0, 1 \ldots . \ . n-r+3)$, of the matrix formed from equations (A) as in 11 will be solutions of (A). Δ_s will be linear in ρ, but $Y'f$ requires the ratios of ρ's to enter in the solutions. We may, therefore, write our invariants

$$\sigma_l = \frac{x_2 - x_l}{x_1 - x_l} \div \frac{x_2 - x_3}{x_1 - x_3},$$

the cross-ratios of the abscissas of four points, and

$$\theta_t = \frac{\Delta_t}{\Delta_0}, \ (t=1 \ \ldots\ldots n-r+3).$$

13. $\boxed{p, \ 2xp + yq, \ x^2 p + xyq}$

From this projective group we may obtain the complete system

$$\sum_1^n i\frac{\partial f}{\partial x_i}=\sum_1^n i\,(2x_i\frac{\partial f}{\partial x_i}+y_i\frac{\partial f}{\partial y_i})=\sum_1^n i\,(x_i^2\frac{\partial f}{\partial x_i}+x_iy_i\frac{\partial f}{\partial y_i})=0.$$

The solutions y_i and $\varphi_j=x_1-x_j$ of the first of these equations introduced in the last give

$$\sum_2^n j(\varphi_j^2\frac{\partial f}{\partial \varphi_j}+y_j\varphi_j\frac{\partial f}{\partial y_j})=0,$$

whose solutions are clearly

$$u_k=\frac{1}{\varphi_2}-\frac{1}{\varphi_k},\quad v_j=\frac{\varphi_j}{y_j},\quad y_1.$$

The second equation is now

$$y_1\frac{\partial f}{\partial y_1}-2\sum_3^n k u_k\frac{\partial f}{\partial u_k}+\sum_2^n j v_j\frac{\partial f}{\partial v_j}=0,$$

with solutions

$$\sigma_l=\frac{u_l}{u_3},\quad \rho_k=\frac{v_k}{v_2},\quad \rho_2=\frac{v_2}{y_1},\quad \rho_1=y_1u_3v_2,\quad (k=3\,.\,.\,.\,.\,.\,n,\ l=4\,.\,.\,.\,.\,.\,n).$$

Our invariants are then

$$\sigma_l=\frac{x_2-x_l}{x_2-x_3}:\frac{x_1-x_3}{x_1-x_l},\quad \rho_k=\frac{x_1-x_k}{x_1-x_2}:\frac{y_k}{y_2},\quad \rho_2=\frac{x_1-x_2}{y_1-y_2},\quad \rho_1=\frac{x_3-x_2}{x_3-x_1}:\frac{y_2}{y_1},$$

whose geometric significance is apparent.

14. $\boxed{yq,\quad p,\quad xp,\quad x^2p+xyq}$

This four-parameter projective group yields the complete system

$$\sum_1^n iy_i\frac{\partial f}{\partial y_i}=\sum_1^n i\frac{\partial f}{\partial x_i}=\sum_1^n ix_i\frac{\partial f}{\partial x_i}=\sum_1^n i(x_i^2\frac{\partial f}{\partial x_i}+x_iy_i\frac{\partial f}{\partial y_i})=0.$$

Here, we introduce the solutions,

$$\varphi_j=x_1-x_j,\quad \psi_j=\frac{y_j}{y_1},$$

of the first two equations in the last two as new variables, and have

$$\sum_1^n j\,\varphi_j\frac{\partial f}{\partial \varphi_j}=\sum_2^n j(\varphi_j^2\frac{\partial f}{\partial \varphi_j}+\varphi_j\psi_j\frac{\partial f}{\partial \psi_j})=0.$$

The last of these new equations is satisfied by

$$\sigma_k=\frac{1}{\varphi_1}-\frac{1}{\varphi_k},\quad \rho_j=\frac{\varphi_j}{\psi_j},$$

while the first becomes

$$\sum_3^n k\sigma_k\frac{\partial f}{\partial \sigma_k}-\sum_2^n j\rho_j\frac{\partial f}{\partial \rho_j}=0.$$

The invariants of this group are, therefore,

$$u_l=\frac{\sigma_l}{\sigma_3}=\frac{x_2-x_l}{x_2-x_3}:\frac{x_1-x_3}{x_1-x_l},\ (l=4\ldots\ldots n).$$

$$v_k=\frac{\rho_k}{\rho_2}=\frac{x_1-x_k}{x_1-x_2}:\frac{y_k}{y_2},\ (k=3\ldots\ldots n),$$

$$v_2=\sigma_3\,\rho_2=\frac{x_2-x_3}{x_1-x_3}:\frac{y_2}{y_1}.$$

§ 3.

Invariants of Such Imprimitive Groups as Leave Unchanged More Than One Family of ∞^1 Curves.

The groups of this section are classified according as they leave invariant:

(A) Two families of ∞^1 curves : x=constant, y=constant.

(B) A single infinity of families of ∞^1 curves : $ax+by$=constant.

(C) ∞^∞ of families of ∞^1 curves : $\varphi(x)+\psi(y)$=constant.

The calculations for the invariants of the remaining groups will be much abbreviated. The subscripts i, j, k, l, will be used to denote the series of numbers running from 1, 2, 3, 4, respectively, to n.

(A) *The families of curves x=constant, y=constant, remain invariant.*

15. $\boxed{q,\quad yq}$

The differential equations

$$\Sigma i\frac{\partial f}{\partial y_i}=\Sigma i\, y_i\frac{\partial f}{\partial y}=0,$$

belonging to this group, evidently leave invariant

$$X_i \text{ and } \psi_k=(y_1-y_k):(y_1-y_2).$$

16. $\boxed{q,\quad yq,\quad y^2q}$

This is the general projective group in one variable, and leaves invariant x_i, and the cross-ratios of any four ordinates. We may determine these cross-ratios by substituting ψ_k of 15 in

$$\Sigma i y_i^2\frac{\partial f}{\partial y_i}=0,$$

giving

$$\Sigma k\psi_k(\psi_k-1)\frac{\partial f}{\partial \psi_k}=0,$$

and integrating,

$$\rho_l=\frac{\psi_l-1}{\psi_l}:\frac{\psi_3-1}{\psi_3}=\frac{y_2-y_l}{y_2-y_3}:\frac{y_1-y_l}{y_1-y_2}.$$

17. $\boxed{q,\quad yq,\quad p}$

This group evidently leaves invariant

$$\psi_k=(y_1-y_k):(y_1-y_2), \text{ and } \varphi_j=x_1-x_j.$$

18. $\boxed{q, \quad yq, \quad y^2q, \quad p}$

The invariants for this group are clearly

$$\rho_l=\frac{y_2-y_l}{y_2-y_3}:\frac{y_1-y_l}{y_1-y_3}, \text{ as in 16 above, and}$$

$$\varphi_j=x_1-x_j, \text{ as in 17.}$$

19. $\boxed{q, \quad p, \quad xp+cyq}$

The solutions of the complete system corresponding to this group must be functions of $\psi_j=y_1-y_j$, $\varphi_j=x_1-x_j$, as shown by the first two members. These solutions, substituted in the last equation, give

$$\Sigma j\varphi_j\frac{\partial f}{\partial\varphi_j}+c\Sigma j\psi_j\frac{\partial f}{\partial\psi_j}=0,$$

with solutions

$$u_k=\frac{x_1-x_k}{x_1-x_2}, \quad v_k=\frac{y_1-y_k}{y_1-y_2}, \quad \sigma=\frac{(x_1-x_2)^c}{y_1-y_2}.$$

20. $\boxed{q, \quad yq, \quad p, \quad xp}$

The invariants here are

$$u_k=\frac{x_1-x_k}{x_1-x_2}, \quad v_k=\frac{y_1-y_k}{y_1-y_2}.$$

21. $\boxed{q, \quad yq, \quad y^2q, \quad p, \quad xp}$

By this group

$$\rho_l=\frac{y_2-y_l}{y_2-y_3}:\frac{y_1-y_l}{y_1-y_3}, \quad u_k=\frac{x_1-x_k}{x_1-x_2}$$

remain invariant.

22. $\boxed{q, \quad yq, \quad y^2q, \quad p, \quad xp, \quad x^2p}$

This six-parameter group leaves invariant the cross-ratios of any four abscissas and ordinates :

$$\rho_l=\frac{y_2-y_l}{y_2-y_3}:\frac{y_1-y_l}{y_1-y_3}, \quad \sigma_l=\frac{x_2-x_l}{x_2-x_3}:\frac{x_1-x_l}{x_1-x_3}.$$

23. $\boxed{p+q, \quad xp+yq, \quad x^2p+y^2q}$

$$\Sigma\left(\frac{\partial f}{\partial x_i}+\frac{\partial f}{\partial y_i}\right)=\Sigma\left(x_i\frac{\partial f}{\partial x_i}+y_i\frac{\partial f}{\partial y_i}\right)=\Sigma\left(x_i^2\frac{\partial f}{\partial x_i}+y_i^2\frac{\partial f}{\partial y_i}\right)=0.$$

The solutions

$$\varphi_j = x_1 - x_j,\quad \psi_j = y_1 - y_j,\quad \sigma = x_1 - y_1$$

of the first equation, as new variables, give

$$\Sigma\left(\varphi_j \frac{\partial f}{\partial \varphi_j} + \psi_j \frac{\partial f}{\partial \psi_j}\right) + \sigma \frac{\partial f}{\partial \sigma} = \Sigma\left(\varphi_j^2 \frac{\partial f}{\partial \varphi_j} + \psi_j^2 \frac{\partial f}{\partial \psi_j}\right) + \sigma^2 \frac{\partial f}{\partial \sigma} = 0.$$

Selecting as solutions of the first of these new equations

$$u_k = \frac{\varphi_k}{\varphi_2},\quad v_k = \frac{\psi_k}{\psi_2},\quad \sigma_1 = \frac{\varphi_2}{\sigma},\quad \sigma_2 = \frac{\psi_2}{\sigma},$$

we have

$$\Sigma\left(u_k(1-u_k)\frac{\partial f}{\partial u_k} + v_k(1-v_k)\frac{\partial f}{\partial v_k}\right) + \sigma_1(1-\sigma_1)\frac{\partial f}{\partial \sigma_1} + \sigma_2(1-\sigma_2)\frac{\partial f}{\partial \sigma_2} = 0.$$

Hence, our invariants may be selected as the cross-ratios

$$\sigma_l = \frac{u_l(1-u_3)}{u_3(1-u_l)} = \frac{x_2 - x_l}{x_2 - x_3} : \frac{x_1 - x_l}{x_1 - x_3},\quad \rho_l = \frac{v_l(1-v_3)}{v_3(1-v_l)} = \frac{y_2 - y_l}{y_2 - y_3} : \frac{y_1 - y_l}{y_1 - y_3},$$

and the ratio

$$t = \frac{\sigma_1(1-\sigma_2)}{\sigma_2(1-\sigma_1)} = \frac{x_1 - x_2}{y_1 - y_2} : \frac{y_1 - x_2}{x_1 - y_2}.$$

(B) *All families of curves of the form* $ax + by = \text{constant}$ *remain invariant.*

24. $\quad q,\quad xp + yq$

The first transformation of this group shows the differences $\psi_j = y_1 - y_j$ to enter in the final solutions. Hence the invariants may be written:

$$\rho_j = \frac{x_j}{x_1},\quad u_k = \frac{y_1 - y_k}{y_1 - y_2},\quad \sigma = \frac{y_1 - y_2}{x_1}.$$

25. $\quad p,\quad q$

This group of translations leaves invariant

$$\varphi_j = x_1 - x_j,\quad \psi_j = y_1 - y_j.$$

26. $\quad p,\quad q,\quad xp + yq$

By this group any homogeneous function of the differences $x_1 - x_j$, $y_1 - y_j$ will remain invariant.

$$u_k = \frac{x_1 - x_k}{x_1 - x_2},\quad v_k = \frac{y_1 - y_k}{y_1 - y_2},\quad \sigma = \frac{x_1 - x_2}{y_1 - y_2}.$$

(C) *The totality of curves* $\varphi(x) + \psi(y) = \text{constant}$ *remains invariant.*

27. $\quad q$

This group is the only one of the class, and evidently leaves invariant the abscissas x_i, and the differences

$$\psi_j = y_1 - y_j.$$

FEOREMA.

Hay numeros cuyo cuadrado se compond de la suma de una serie de cubos. j Cuales son estos numeros ? Fomernos la serie de los numeros naturales principrando por uno y vayamos hasta un numero calquiera, por ejemplo, hasta dies, asi.

1	2	3	4	5	6	7	8	9	10

Debajo de estos numeros pongamos sus cubos asi

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000

Sumemos estos cubos de manera que el primero venga á colo carse bajo el primer numero, la suma del primero y del segundo, bajo el segundo cubo : la suma, del primero segundo y tercero, bajo el tercero, y asi sucesuamente, tendremos

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000
1	9	36	100	225	441	784	1296	2025	3025

y encontraremos que la suma de dichos cubos son cuadrados y sus raices son

	1	2	3	4	5	6	7	8	9	10
	1	8	27	64	125	216	343	512	729	1000
	1	9	36	100	225	441	784	1296	2025	3025
Raiz	1	3	6	10	15	21	28	36	45	55

Consideremos estas y raices veremos que son la suma de los térnimos de progresiones aritmeticas principrando por uno y segundo por los numeros naturales hasta cualquier numero que se desee y desde luego podremos formular el tevrema del modo siguiente.

Los cuadrados de los numeros que indican la suma de los términos deprogresiones aritmeticas, principrando por uno y siguiendo por los numeros naturales ; dichos cuadrados decimos, son iguales á la suma de los cubos de todos los numeros que entran en dicha progresion : ver : gr :

300. Es la suma de los terminos de una progresion aritmetica principrando por uno y acabando por veinte y cuatro y

300 elevado á la segunda potencia $=1^3+2^3+3^3+4^3 \ldots\ldots +24^3$.

J. M. Monsanto.

TRANSLATION.

Theorem.

There are numbers whose square is composed of the sum of a series of cubes. Which are these numbers?

Let us take the series of the primary numbers beginning with one and let us proceed as far as any number, for example, as far as ten, thus

1	2	3	4	5	6	7	8	9	10

Under these numbers let us place their cubes thus

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000

Let us add these cubes in such a way that the first comes to be placed under the first number, the sum of the first and the second under the second cube ; the sum of the first, second and third, under the third cube, and so on, we shall have

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000
1	9	36	100	225	441	784	1296	2025	3025

and we shall find that the sum of the said cubes are squares, and their roots are

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000
1	9	36	100	225	441	784	1296	2025	3025
1	3	6	10	15	21	28	36	45	55

Let us consider these roots and we shall see that they are the sum of the terms of an arithmetical progression beginning with one and continuing through the primary numbers as far as any number desired, and therefore we can formulate the theorem in the following manner :

The squares of the numbers that indicate the sum of the terms of an arithmetical progression, beginning with one and continuing through the primary numbers, said squares, we repeat, are equal to the sum of the cubes of all the numbers that enter in the said progression : for example 300^2. 300^2 is the sum of the terms of an arithmetical progression beginning with 1 and ending with 24, and 300 raised to the second power$=1^3+2^3+3^3+4^3\ldots\ldots+24^3$.

J. M. MONSANTO.

NOTE.—This rare bit of original mathematical work by J. M. Monsanto, with its translation, was sent to us by Dr. Halsted. ED. F.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

100. Proposed by CHAS. C. CROSS, Libertytown, Md.

I bought stock at 4% discount, and sold it at $2\frac{1}{2}\%$ premium, paying a brokerage in both cases of $\frac{1}{4}\%$. If my net profits were $130, what was my investment? (Solve by Arithmetic).

I. Solution by W. F. BRADBURY, A.M., Head Master Cambridge Latin School, Cambridge, Mass., and M. E. GRABER, Tiffin, Ohio.

$4\%+2\frac{1}{2}\%-\frac{1}{2}\%$ (brokerage) $=6\%$.

That is, he made $6 on every $100 he invested. $130\div6=21\frac{2}{3}$.

He bought $21\frac{2}{3}$ shares of $100 each, or $\$2166\frac{2}{3}$ worth of stock.

But it cost him $3\frac{3}{4}\%$ below par.

$\$2166\frac{2}{3}\times0.96\frac{1}{4}=\$2085.415$.

Solved in a similar manner by *P. S. BERG* and *W. H. DRANE*.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

"In both cases" is a dubious expression. If the brokerage in the two transactions was *together* $\frac{1}{4}\%$, then the net gain on a dollar $=.04+.02\frac{1}{2}-.00\frac{1}{4}=.06\frac{1}{4}$, and the investment $=\$130\div.06\frac{1}{4}=\$2080$.

But if the brokerage in *each* transaction was $\frac{1}{4}\%$, then the net gain on a dollar $=.04+.02\frac{1}{2}-.00\frac{1}{2}=.06$; and the investment $=\$130\div.06=\$2166.66\frac{2}{3}$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and ALOIS F. KOVARIK, Instructor in Mathematics, Decorah Institute, Decorah, Iowa.

Let $100 = 1 share.

$4\%-\frac{1}{4}\%=3\frac{3}{4}\%$; $100\%+2\frac{1}{2}\%-\frac{1}{4}\%=102\frac{1}{4}\%$. $100\%-3\frac{3}{4}\%=96\frac{1}{4}\%$.

$\$100\div.96\frac{1}{4}=\$103.896$.

$\$103.896\times1.02\frac{1}{4}=\$106.23366$.

$\$106.23366-\$100=\$6.23366$, gain on one share.

$\$130\div\$6.23366=20.85452$ shares.

$20.85452\times\$100=\$2085.452$, amount invested.

101. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

A man gained $m=3\%$ on his money, in July; and, in August, lost $n=2\%$. What per cent. of his money July 1st is his money September 1st?

I. Solution by P. S. BERG, Principal of Schools, Larimore, N. D., and JOHN F. TRAVIS, Student in Ohio State University, Columbus, Ohio.

If 100% is his money on July 1st, then 103% is his money August 1st, and 98% of 103% or 100.94% is his money on September 1st.

Therefore his money September 1st is 100.94% of his money July 1st.

II. Solution by J. M. COLAW, A. M., Monterey, Va.; W. F. BRADBURY, A. M., Head Master Cambridge Latin School, Cambridge, Mass.; M. A. GRUBER, A. M., War Department, Washington, D. C.; WALTER HUGH DRANE, Graduate Student, Harvard University; and J. O. MAHONEY, B. E., M. Sc., Master of Mathematics and Science, Carthage Graded and High School, Carthage, Texas.

If p is his principal on July 1st, then $\frac{p(100+m)}{100}$=his money on Aug. 1st.

Also, $\frac{p(100+m)}{100}-\frac{pn(100+m)}{10000}$, or $\frac{p(100-n)(100+m)}{10000}$=his money on September 1st.

$$\therefore \frac{p(100-n)(100+m)}{10000}\div p=\frac{(100-n)(100+m)}{10000},=\left[\frac{(100-n)(100+m)}{100}\right]\%.$$

Substituting numerical values for m and n, respectively, we get $\frac{98\times 103}{10000}=1.0094,=100.94\%$.

The result is independent of the value of p

Also solved by *G. B. M. ZERR*.

ALGEBRA.

88. Proposed by E. S. LOOMIS, Ph. D., Professor of Mathematics in Cleveland West High School, Berea, O.

(1) The Indemnity Savings and Loan Company made two loans of \$1000 each to "A", one of its borrowers, under the following terms: In the first loan "A" agrees to cancel the \$1000 by making 120 payments of \$13.50, the first payment to be considered as made on the first of the month in which the loan is made, and the 119 subsequent payments to be made on the first of each subsequent month; in the second loan "A" agrees to cancel the \$1000 by making 120 payments of \$13.50, the first payment being made on the first of the month following the loan, and the 119 subsequent payments being made on first of the subsequent months. Does the Company sustain any loss in earnings by the second loan over the first loan, and if so how much, and when is (or are) this loss (or these losses) sustained, the rate of interest in each case being considered as $10\frac{1}{2}\%$ per annum ?

(2) Deduce a formula for each case of proposition (1) by means of which one can find the balance of the loan uncancelled at the end of *any* month, *if* the loan is fully cancelled in 120 payments.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

(1) Let P=principal, r=rate of interest, n=number of payments, x=each payment. Then it has been shown in several previous problems that

$$x=\frac{Pr(1+r)^n}{(1+r)^n-1}.$$

In the first case "A" gets at once \$1000.00—\$13,50=\$986.50, and pays this amount in 119 monthly installments.

$\therefore P=986.50$, $r=10\frac{1}{2}\div 12=0.875\%$, $n=119$.

$$\therefore x=\frac{986.50\times .00875(1.00875)^{119}}{(1.00875)^{119}-1}.$$

x=\$13.375 or \$0.125 less than \$13.50.

In the second case $P=1000$, $r=.00875$, $n=120$.

$\therefore x=\dfrac{1000\times .00875(1.00875)^{120}}{(1.00875)^{120}-1}$.

$\therefore x=\$13.494$.

In the second case the true interest is realized. $\therefore$ 10½ per cent. is charged. In the first case more than 10½ per cent. is charged. $\therefore$ In the first case the borrower by paying \$13.50 pays 12½ cents more each payment than if he paid 10½ per cent. interest per annum.

(2) Pr=interest. $\therefore x-Pr$=principal paid at the first payment.

$P(1+r)-x$=new principal; $Pr(1+r)-rx$=second interest.

$\therefore x(1+r)-Pr(1+r)=(x-Pr)(1+r)$=principal paid at second payment.

Similarly, $(x-Pr)(1+r)^{m-1}$=principal paid at the mth payment.

$\therefore x-Pr$=first term, $1+r$=ratio, m=number of terms.

$\therefore S=\dfrac{(x-Pr)[(1+r)^m-1]}{r}$.

Unpaid principal at the end of m payments is $P-S$.

Let $x=13.50$, $P=1000$, $r=.00875$, $m=60$.

$\therefore S=\$372.72$.

$\therefore P-S=\$1000-\$372.72=\$627.28$ unpaid.

II. **Partial Solution by A. H. BELL, Hillsboro, Ill.**

(1) The first loan of \$1000 for 10 years at 10½ per cent. amounts to \$2714.06......(1).

The second loan of \$986.50 for $9\frac{11}{12}$ years at 10½ per cent. amounts to \$2655.24......(2).

Let the common form of the amount of an annuity be:

$$\text{Amount annuity}=A\left(\frac{R'^n-1}{r'}\right)\ldots\ldots(3),$$

in which $A=\$13.50$, n=number of payments, R'=the amount of \$1.00 for one month at 10½ per cent.=\$1.00875, r'=interest for one month at 10½ per cent.= .00875.

$\therefore$ The amount of the annuity \$13.50 for 120 months is \$2847.26.. .(4); the amount of the annuity \$13.50 for 119 months is \$2809.18 ....(5).

(4)—(1)=\$133.20 is the profit in the first transaction. ....(6);

(5)—(2)=\$153.94 is the profit in the second transaction......(7).

Since the second loan of \$1000 is immediately reduced by the first payment of \$13.50, no comparison can be made with the first loan, for the second actual loan of \$986.50 pays a greater amount of annuity on the dollar and consequently shows a greater amount earned as seen in (7).

III. **Solution by FREMONT CRANE, Sand Coulee, Mont.**

(a) First payment could draw interest for 120 months; last payment could draw interest for 1 month. $\therefore$ average time=60½ months.

(b) First payment would draw interest for 119 months; last payment would draw interest for 0 months. $\therefore$ average time is 59½ months.

$\therefore$ (a) would be the better investment.

$120\times\$13.50=\$1620$; $\$1620\times.10\frac{1}{2}\times\frac{1}{12}=\$14.175$=profit of first investment over second.

2. (a) Let $\$13.50=p$, $.00875=r$, $\$1000=a$; then $(a-p)$=amount due first of first month ; $(a-p)(1+r)-p$=amount due first of second month ; $[(a-p)(1+r-p)](1+r)-p$=amount due first of third month ; $\{[(a-p)(1+r)-p)](1+r)-p\}(1+r)-p$=amount due first of fourth month.

$\therefore$ for any month, as 10,

$\{[(a-p)(1+r)-p](1+r)-p](1+r)-p\}$ $[(a-p)(1+r)-p)(1+r)-p^7(1+r)-p]$.

2. $a(1+r)-p$ at end of first month ; $[a(1+r)-p](1+r)-p$ at end of second month ; $\{[a(1+r)-p](1+r)-p\}(1+r)-p$ at end of third month. At end of tenth month, $\langle\{[a(1+r)-p](1+r)-p\}^8(1+r)-p\rangle$.

The solution by Dr. Zerr seems to us to be a proper disposition of the problem, but we publish the other solutions for comparison.

Reply to Note of Dr. Drummond on No. 78.—Its one *real* point is, that my solution, on page 43, line 5, omits to denote $-xy=-6$, as $-x_1y_1=-6$, (which please correct). Thus result $x^4-8x^2+16=0$; $y^4-18y^2+81=0$, whose roots are : $x=2$ to $y=3$; $x_a=-2$ to $y_a=-3$, of xy in (I)(II) ; and also $x_1=2_1$ to $y_1=3_1$; $x_2=-2_1$ to $y_2=-3_1$ (by last two clauses on page 44), the factors of $-x_1y_1=-6_1$ in parity to $x_{11}y_{11}=35$......(D) on page 44. Or if the objector insist,—of $x_1y_1=6_1$ and $x_ay_a=6_a$ reading then $xy=6$ and $x_ay_a=6_a$. No other error is specified or shown. As the equations cited do not occur in my solution, —for his $x^2=36$ is quadratic while $x^2y^2=36$ from (I)(II) is bi-quadratic ; I need only say that the roots of his $x^4=1296$ are 6 ; -6, $6\sqrt{(-1)}$; $-6\sqrt{(-1)}$, not ±6 ; ∓6. Roots $x^4=16$ are 2 ; -2 ; $2\sqrt{(-1)}$; $-2\sqrt{(-1)}$ and not ±2 ; ∓2 as he mistakes me to mean. J. M. Boorman.

GEOMETRY.

99. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Find the locus of the vertices of all right cones which have the same given ellipse as a base.

Solution by the PROPOSER.

Any generator, $x=a'z+\alpha$..... (1), $y=b'z+\beta$......(2), must satisfy

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$......(3), and $z=0$......(4), giving $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1$......(5).

If the vertex be (x', y', z'), (1) and (2) must be satisfied, and we have $x'=a'z'+\alpha$......(6), $y'=b'z'+\beta$ .....(7).

Eliminating a', b', α, β from (1), (2), (6), (7),

$$\frac{1}{a^2}(z'x-x'z)^2+\frac{1}{b^2}(z'y-y'z)^2=(z'-z)^2 \text{.....} \quad (8).$$

the equation to a cone having (x', y', z') for vertex, and (3) for base.

$$(8) \text{ is } \frac{z'^2}{a^2}x^2+\frac{z'^2}{b^2}y^2+\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)z^2-\frac{2y'z'}{b^2}yz-\frac{2x'z'}{a^2}xz+2z'z-z'^2=0 \quad ..(9).$$

The conditions for (9) to be a cone of revolution are $x'=0$,

$$\frac{z'^2}{a^2}-\frac{y'^2}{b^2-a^2}=1 \ldots\ldots(10),$$

an hyperbola for the required locus.

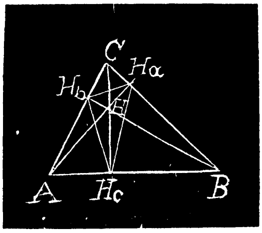
Also solved by *G. B. M. ZERR*.

100. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

O, O_1, O_2, O_3 are the centers of the inscribed and three escribed circles of a triangle ABC. Prove $AO.AO_1.AO_2.AO_3=AB^2.AC^2$.

I. Solution by the PROPOSER.

Consider the ex-central triangle ABC as the original triangle. Then $H_aH_bH_c$ is the pedal triangle, and the incenter O becomes the orthocenter H.



Hence we have to prove $AH_c\times BH_c\times CH_c\times HH_c=H_aH_c^2\times H_bH_c^2$.

We readily find by trigonometry that

$$AH_c=\frac{b^2+c^2-a^2}{2c},\ BH_c=\frac{a^2+c^2-b^2}{2c},$$

$$CH_c=\frac{1}{2c}\frac{[4a^2c^2-(a^2+c^2-b^2)^2]}{}=\frac{2\triangle}{c},\quad HH_c=\frac{(b^2+c^2-a^2)(a_2+c^2-b^2)}{8c\triangle},$$

$$H_aH_c=\frac{b(a^2+c^2-b^2)^2}{2ac},\ H_bH_c=\frac{a(b^2+c^2-a^2)}{2bc}.$$

Substituting in the problem we have

$$\frac{b^2+c^2-a^2}{2c}\times\frac{a^2+c^2-b^2}{2c}\times\frac{2\triangle}{c}\times\frac{(b^2+c^2-a^2)(a^2+c^2-b^2)}{8c\triangle}$$

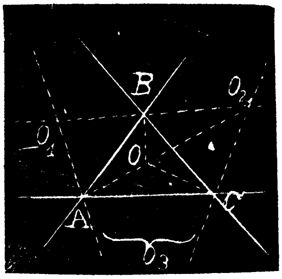
$$=\frac{b^2(a^2+c^2-b^2)^2}{4a^2c^2}\times\frac{a^2(b^2+c^2-a^2)^2}{4b^2c^2}.$$

Since these equations cancel, the proposition is proved.

Mr. Cross should have been credited with solutions of problems 96 and 98 in Geometry, and 100 and 101 in Arithmetic.

II. Solution by WALTER H. DRANE, Graduate Student at Harvard University, and J. SCHEFFER, A. M., Hagerstown, Md.

Let ABC be the given triangle, O, O_1, O_2, O_3, the centers of the inscribed and the three escribed circles. Then O_1BO_2, O_1AO_3, O_2CO_3, and AOO_2 are straight lines; also OB, OA, OC are each perpendicular to O_1BO_2, O_1AO_3, and O_2CO_3, respectively. In triangles AOC and BAO_2, $\angle OAC=BAO_2$ and $\angle OCA=\angle BO_2A$ since we have $\angle OCA=90^\circ-\angle ACO_3=90^\circ-\frac{1}{2}(\angle CAB+\angle CBA)=90^\circ-(\angle OAB+OBA)=90^\circ-[180^\circ-(\angle O_1AB+\angle O_1BA)]=90^\circ-\angle O=\angle O_1O_2A$.



$\because$ triangles AOC and ABO_2 are similar, and we have

$$AO:AB::AC:AO_2\ \dots\dots\dots(1).$$

Again in triangles O_1BA and AO_3C, $\angle O=\angle ACO_3$ and $\angle O_3AC=\angle O_1AB$. Hence triangles O_1BA and AO_3C are similar, and we have,

$$AO_1:AC::AB:AO_3\ \dots\dots\dots\dots\dots(2).$$

Multiplying (1) by (2) $AO.AO_1:AB.BC:AB.AC:AO_2.AO_3$.

$\because AO.AO_1.AO_2.AO_3=AB^2.AC^2$. Q. E. D.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

In the figure of the last solution, draw the lines OD and O_1D_1 perpendicular to AC. Then $AO=\sqrt{(OD^2+AD^2)}=\sqrt{(r^2+r^2\cot^2\frac{1}{2}A)}=r\text{cosec}\frac{1}{2}A$.

Similarly, $AO_1=r_1\text{cosec}\frac{1}{2}A$.

$AO_2=\sqrt{(OO_2^2-AO^2)}=AO\sqrt{[(OO_2^2/AO^2)-1]}=AO\cot\frac{1}{2}C$.

Similarly, $AO_3=AO\cot\frac{1}{2}B$.

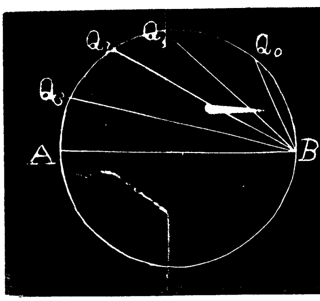
$$\begin{aligned}\therefore AO.AO_1.AO_2.AO_3&=r^3r_1\text{cosec}^4\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C\\&=(s-a)(s-b)(s-c)s\text{cosec}^4\tfrac{1}{2}A\tan^2\tfrac{1}{2}A\\&=s(s-a)(s-b)(s-c)/\sin^2\tfrac{1}{2}A\cos^2\tfrac{1}{2}A\\&=4s(s-a)(s-b)(s-c)/\sin^2A=4S^2/\sin^2A.\end{aligned}$$

But $\sin A=2S/bc=2S/AB.AC$. $\therefore AO.AO_1.AO_2.AO_3=AB^2.AC^2$.

Also solved by *ELMER SCHUYLER*.

101. Proposed by E.W. MORRELL, A.M., Late Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

AB is the diameter of a circle and Q_0 any point on the circumference; Q_1, Q_2, Q_3.... are the points of bisection of the arcs AQ_0, AQ_1, AQ_2.... Prove that BQ_1, BQ_2, BQ_3 ....$BQ_n=OA^n.(AQ_0/AQ_n)$.



Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J. SCHEFFER, A.M., Hagerstown, Md., and ELMER SCHUYLER, High Bridge, N. J.

Let O be the center of the circle.

$\angle ABQ_0=\theta$.

$\therefore BQ_1=AB\cos\frac{1}{2}\theta$, $BQ_2=AB\cos(\theta/2^2)$.

$BQ_3=AB\cos(\theta/2^3)$, $BQ_n=AB\cos(\theta/2^n)$.

$$AQ_0=AB\sin\theta=2AB\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta=2^2AB\sin(\theta/2^2)\cos(\theta/2^2)\cos\tfrac{1}{2}\theta$$
$$=2^nAB\sin(\theta/2^n)\cos\tfrac{1}{2}\theta\cos(\theta/2^2)\ldots\cos(\theta/2^n).$$
$$AQ_n=AB\sin(\theta/2^n).$$
$$\therefore\ (AQ_0/AQ_n)=2^n\cos\tfrac{1}{2}\theta\cos(\theta/2^2)\ldots\cos(\theta/2^n).$$
$$BQ_1.BQ_2.BQ_3\ldots BQ_n=(AB^n)\cos\tfrac{1}{2}\theta\cos(\theta/2^2)\ldots\cos(\theta/2^n)$$
$$=(\tfrac{1}{2}AB)^n(AQ_0/AQ_n)=(AO)^n(AQ_0/AQ_n).$$

CALCULUS.

78. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Investigate value of $\left(\frac{\tan x}{x}\right)^{1/x^n}$ where x is 0 and n has consecutive values 1, 2, 3, 4, ... Is there any law governing the different results? When $n=1$, result is 1; when $n=2$, result is $e^{\frac{1}{3}}$; $n=3$, gives ∞, etc.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

$$\left(\frac{\tan x}{x}\right)^{1/x^n}=e^{(1/x^n)\log[(\tan x)/x]}=y.$$

Limit of $\dfrac{\log\tan x-\log x}{x^n}$ = limit of $\dfrac{\cot x\sec^2x-(1/x)}{nx^{n-1}}$ = limit of $\dfrac{2x-\sin 2x}{nx^n\sin 2x}$,

but $\sin 2x=2x-\dfrac{8x^3}{6}+\dfrac{32x^5}{120}-\dfrac{128x^7}{5040}$, etc., $=2x-\dfrac{4x^3}{3}+\dfrac{4x^5}{15}-\dfrac{4x^7}{45}+$, etc.

$$\therefore\ \frac{2x-\sin 2x}{nx^n\sin 2x}=\frac{2x-2x+\dfrac{4x^3}{3}-\dfrac{4x^5}{15}+\dfrac{4x^7}{45}-,\text{ etc.}}{nx^{n+1}(2-\dfrac{4x^2}{3}+\dfrac{4x^4}{15}-\dfrac{4x^6}{45}+,\text{ etc.}}=\frac{30-6x^2+2x^4}{nx^{n-2}(45-30x^2+6x^4)},\text{ ap-}$$

$$\text{proximately, }=\frac{2}{3nx^{n-2}}+\frac{14}{45nx^{n-4}}+,\text{ etc., }=S.$$

When $n=1$, $S=0$ for $x=0$.
When $n=2$, $S=\frac{1}{3}$ for $x=0$.
When $n=3$, 4, 5, etc., $S=\infty$ for $x=0$.
$\therefore$ When $n=1$, $y=e^0=1$.
When $n=2$, $y=e^{\frac{1}{3}}$.
When $n=3$, 4, 5, etc., $y=e^{\infty}=\infty$.

Also solved by *ELMER SCHUYLER*, whose solution has been accidentally misplaced, and hence does not appear in this issue.

79. Proposed by GEORGE LILLEY, Ph.D., LL. D., Professor of Mathematics, University of Oregon, Eugene, Oregon.

Find the area included between $y=\sin^\pi x+\cos^e x$; $y=\pi e(\sin^\pi x\cos^e x)$ and the length of its boundary, true to six decimal places, when $\pi=3.14159$, $e=2.7182$.

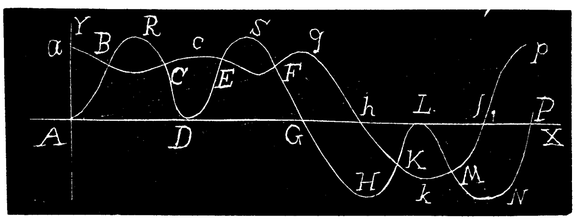
Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

When $x=0^\circ$, $\sin^\pi x=0$, $\cos^e x=1.0000$.
When $x=30^\circ$, $\sin^\pi x=.1133$, $\cos^e x=.6764$.
When $x=45^\circ$, $\sin^\pi x=.3366$, $\cos^e x=.3899$.
When $x=60^\circ$, $\sin^\pi x=.8006$, $\cos^e x=.1520$.
When $x=90^\circ$, $\sin^\pi x=1.0000$, $\cos^e x=0$.

Since e is even, $\cos^e x$ is always positive ; hence the following coördinates :

	$y=\sin^\pi x+\cos^e x$	$y=\pi e(\sin^\pi x\cos^e x)$
x	y	y
0° = .0	1.0000	0
30° = .5236	.7897	.6541
45° = .7853	.7265	1.1207
60° = 1.0471	.9526	1.0392
90° = 1.5707	1.0000	0
120° = 2.0942	.9526	1 0392
135° = 2.3560	.7265	1.1207
150° = 2.6178	.7897	.6541
180° = 3.1414	1.0000	0
210° = 3.6650	.5631	−.6541
225° = 3.9268	.0533	−1.1207
240° = 4.1886	−.6480	−1.0392
270° = 4.7121	−1.0000	0
300° = 5.2360	−.6480	−1.0392
315° = 5.4978	.0533	−1.1207
330° = 5.7596	.5631	−.6541
360° = 6.2828	1.0000	0

The curves drawn from these coördinates are somewhat like the figure, $ABRCDESFGHKLMNP$ corresponding to $y=\pi e(\sin^\pi x\cos^e x)$; $aBCEFghKkMlp$ corresponding to $y=\sin^\pi x+\cos^e x$.



The curves are indefinite in length. The areas included between the curves from $x=0^\circ$ to $x=360^\circ$ are $AaB+BRC+CDEc+ESF+FGHKhg+LMkK+PNMlp$. The length of the boundary is the whole length of both curves. The whole area common to both curves is infinite. The area above is but the area for one revolution. Area BRC=area ESF,

area AaB + area $PNMlp$ = area $FGHKhg$. The intersections are $x=32^\circ\ 48'$, $x=61^\circ$, $x=119^\circ$, $x=147^\circ\ 12'$, $x=244^\circ\ 22'$, $x=295^\circ\ 38'$. The integrations are exceedingly tedious, but can be performed. If 3.1416 had been used for π the curves for one revolution would have consisted of four parts each equal to $aBCc$ and $ABRCD$.

MECHANICS.

Criticism on Professor Zerr's Solution of Problem 67, Mechanics, by J. M. ARNOLD, Crompton, R. I.

I wish to take exception to Professor Zerr's solution of No. 63 Mechanics, in the May number. The preliminary reasoning and the diagram are correct, but when he proceeds to find the required angles he commences with the assumption "The $\angle ABC = \angle CDE$ and the $\angle BAC = \angle CED$." This is wrong as it can be easily shown that these angles are not equal. Therefore his result must be in error. I have not had time to solve the problem correctly, but I think it leads to very complicated equations.

70. Proposed by CHARLES E. MEYERS, Canton, Ohio.

A homogeneous sphere, radius r, having an angular velocity ω, gradually contracts by cooling. What will be the angular velocity at the instant the radius becomes $\frac{1}{2}r$?

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let m=the constant mass ; r, $\frac{1}{2}r$ the original and final radii ; ω', the required angular velocity ; k, k' the radii of gyration corresponding.

The moment of angular momentum remaining constant,

$$mk^2\omega = mk'^2\omega' \ldots\ldots (1).$$

But $k^2=\frac{1}{2}r^2$, $k'^2=\frac{1}{2}(\frac{1}{2}r)^2=\frac{1}{8}r^2$, (1) plainly gives $\omega'=4\omega$.

Also solved in the same manner by *G. B. M. ZERR.*

71. Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.

Three men own a sphere of gold the density of which varies as the square of the distance from the center. If two segments be cut off each one inch from the center of the sphere it will be divided into three parts of equal value. Determine the diameter of the sphere.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let ρ=density=r^2 in this case, a=radius. Then the mass of each segment cut off is

$$M=\int_1^a\int_0^{2\pi}\int_0^{\cos^{-1}(1/a)}\rho r^2\sin\theta d\theta d\phi dr=\int_1^a\int_0^{2\pi}\int_0^{\cos^{-1}(1/a)}r^4\sin\theta dr d\phi d\theta$$

$$=\frac{2\pi}{5a}(a^5-1)(a-1).$$

The mass of the remaining part is

$$M_1=2\int_0^1\int_0^{2\pi}\int_{\cos^{-1}(1/a)}^{\frac{1}{2}\pi} r^4\sin\theta dr d\varphi d\theta=\frac{4\pi}{5a}.$$

But $M=M_1$.

$\therefore \frac{4\pi}{5a}=\frac{2\pi}{5a}(a^6-a^5-a+1)$.

$\therefore a^6-a^5-a-1=0$.

$\therefore a^3-(1/a^3)=a^2+(1/a^2)$, or $t^3-t^2+3t-2=0$ where $t=a-(1/a)$.

Let $t=s+\frac{1}{3}$, $\therefore s^3+\frac{8}{3}s=\frac{29}{27}$.

$\therefore s=.381891$, $t=.715224$.

$a=\frac{1}{2}[t+\sqrt{(4+t^2)}]=1.4196$ inches nearly.

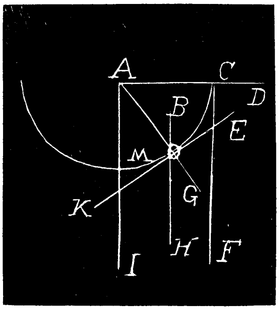
Similarly solved by *J. SCHEFFER*, his answer being 3.7462 inches.

72. Proposed by REV. A. L. GRIDLEY, Pastor of First Congregational Church, Kidder, Mo.

Prove that the motion of a ball falling through the earth influenced by gravity alone would be similar to the motion of a pendulum.

I. Solution by the PROPOSER.

It is an established fact that if a body should fall through an opening through the center of the earth, the net force drawing it toward the center is as the remaining distance to be traversed.



The pendulum ball follows the same law. It is evident that when the rod is horizontal all the force of gravity is exerted upon it to move it along its vertical tangent. When the rod is vertical, there is no net available force to move it from its position.

Making it general, by resolution of forces, the net force of gravity available to move the ball along the tangent EK is, in the figure, equal to the $\measuredangle AMB$. But this angle $=\measuredangle IAM$, as lines AI and BH are parallel and are cut by line AG. But $\measuredangle IAG$ represents the remaining distance for the ball to pass through. Thus the law in both cases is the same.

Inference. 1st. To vibrate in the same time the whole arc—semi-circle—of the pendulum must equal the diameter of the earth.

2d. Vibrating past the center for any distance would require the same time as from surface to surface.

3d. The net available force acting is in inverse proportion to the distance passed through.

Similarly solved by *ALOIS F. KOVARIK*.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; CHARLES C. CROSS, Libertytown, Md.; CHARLES E. MEYERS, Canton, Ohio, and J. SCHEFFER, A. M., Hagerstown, Md.

The equation of motion is $d^2x/dt^2=-gx/R$. Where g=gravity=$32\frac{1}{6}$ feet, R=3963 miles=radius of earth. Multiplying by $2dx$ and integrating, we get $(dx/dt)^2=-gx^2/R+C$.

Let $x=a$ when $dx/dt=0$. Then $C=ga^2/R$.

$\therefore (dx/dt)^2=(g/r)(a^2-x^2)$......(1).

$\therefore dt=\sqrt{(R/g)}.dx/\sqrt{(a^2-x^2)}$.

$\therefore t=\sqrt{(R/g)}\cos^{-1}(x/a)+C_1$.

Since $t=0$ when $x=a$, $C_1=0$.

$\therefore t=\sqrt{(R/g)}\cos^{-1}(x/a)$......(2).

Equation (2) represents simple harmonic motion. From (1), velocity is zero when $x=\pm a$ and a maximum when $x=0$.

When $x=0$, $t=\frac{1}{2}\pi\sqrt{(R/g)}$. This time is the same for any value of a between 0 and R.

$2t$=time of complete vibration$=\pi\sqrt{(R/g)}$.

$2t$=42 minutes, 12 seconds.

Also solved by *J. H. DRUMMOND*.

DIOPHANTINE ANALYSIS.

69. Proposed by JOSIAH H. DRUMMOND, LL. D., Counselor at Law, Portland, Me.

Two right angled triangles have the same base which is a mean proportional between the two perpendiculars: find a general solution, that will give integral values for all the sides of both triangles.

I. Solution by the PROPOSER.

Let x=the base, y=the perpendicular of one triangle, and z=that of the other triangle. Then $x^2=yz$......(1), and $x^2+y^2=\square$......(2), and $x^2+z^2=\square$ .....(3). Substituting yz for x^2 in (2) and (3), we have $zy+y^2=\square$......(4), and $zy+z^2=\square$......(5). Take $y=mz$ and substituting in (4) and (5), and reducing, we have $m+1=\square=$, say, p^2; and $m=p^2-1$.

$m(m+1)=\square$......(6), substituting the value of m in (6), we have $p^2(p^2-1)=\square$. Hence we have to make $p^2-1=\square=$, say, $(q-p)^2$. Hence $p=(q^2+1)/2q$ and $m=[(q^2-1)/2q]^2$; and $y=z[(q^2-1)/2q]^2$. $x=\sqrt{(yz)}=z[(q^2-1)/2q]$; to obtain integral values take $z=4q^2$, and we have $x=2q(q^2-1)$, $z=4q^2$ and hypotenuse$=2q(q^2+1)$, and $x=2q(q^2-1)$, $y=(q^2-1)^2$ and hypotenuse$=(q^2+1)(q^2-1)$, in which q may be any number >1.

If the value of q is an odd number, it is manifest that one fourth of each of the foregoing values will be integral, and we shall have smaller values.

II. Solution by G. B. M. ZERR, A. M.,Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let p, h be the perpendiculars; c, d the hypotenuses; k the common base.

Then $p^2+k^2=c^2$, $h^2+k^2=d^2$, $ph=k^2$.

Let $p=a^3+a^2b$, $h=ab^2+b^3$.

$\therefore$ $k^2=a^2b^2(a+b)^2$.

$\therefore$ $a^2(a+b)^2(a^2+b^2)=c^2$, $b^2(a+b)^2(a^2+b^2)=d^2$.

Let $a=m^2-n^2$, $b=2mn$, $m>n$.

$\therefore$ $p=(m^2-n^2)^2(m^2-n^2+2mn)$,

$h=4m^2n^2(m^2-n^2+2mn)$,

$k=2mn(m^2-n^2)(m^2-n^2+2mn)$,

$c=(m^2-n^2)(m^2-n^2+2mn)(m^2+n^2)$,

$d=2mn(m^2-n^2+2mn)(m^2+n^2)$.

Let $m=2$, $n=1$.

$\therefore$ $p=63$, $h=112$, $k=84$, $c=105$, $d=120$.

III. Solution by **M. A. GRUBER, A. M.**, War Department, Washington, D. C.; **SYLVESTER ROBINS**, North Branch Depot, N. Y., and **CHARLES C. CROSS**, Libertytown, Md.

(a) Take $2mn$, m^2-n^2, and m^2+n^2, the general expressions for the sides of right triangles. Now multiply these sides, respectively, by $r(m^2-n^2)$ and $2mnr$ and we obtain the respective sides of the two required triangles :

(1) $2mnr(m^2-n^2)$, $r(m^2-n^2)^2$, $r(m^2-n^2)(m^2+n^2)$,

(2) $2mnr(m^2-n^2)$, $r(2mn)^2$, $2mnr(m^2+n^2)$,

in which m, n and r may be any integers, $m>n$.

Take $n=r=1$ and $m=2$. Then the sides of the two triangles are, respectively, 12, 9, 15, and 12, 16, 20 ; for $12^2=9\times16$.

$m=2$, $n=1$, $r=3$, gives 36, 27, 45, and 36, 48, 60.

$m=r=3$, $n=2$, gives 180, 75, 195, and 180, 432, 468.

[Gruber, Cross.]

(b) Multiply the sides of any integral right triangle by each leg, or the same number of times each leg, respectively, of the triangle, and the two results will be the respective sides of the two required triangles.

Take the sides 8, 15, 17. Multiply, respectively, by the legs 8 and 15, and we obtain the triangles 120, 64, 136, and 120, 225, 255.

Take the sides 5, 12, 13. Multiply, respectively, by 2×5 and 2×12, and we obtain the triangles 120, 50, 130, and 120, 288, 312.

[Robins, Gruber.]

IV. Solution by **WILLIAM HOOVER, A. M., Ph. D.**, Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Let $2mn$, m^2-n^2, and m^2+n^2; $2m'n'$, $m'^2-n'^2$ and $m'^2+n'^2$ be the pair of right triangles fulfilling the conditions

$$2mn=2m'n' \ldots\ldots (1),$$

$$\text{and } 4m^2n^2=4m'^2n'^2=(m^2-n^2)(m'^2-n'^2) \ldots\ldots (2).$$

From (2), $m'^2-n'^2=\dfrac{4m^2n^2}{m^2-n^2}\ldots\ldots(3)$.

Squaring (3), $m'^4-2m'^2n'^2+n'^4=\dfrac{16m^4n^4}{(m^2-n^2)^2}\ldots\ldots(4)$.

We have $4m'^2n'^2=4m^2n^2$......(5).

Adding (4) and (5), $(m'^2+n'^2)^2=4m^2n^2\left(\frac{4m^2n^2}{(m^2-n^2)^2}+1\right)$

$$\text{or } m'^2+n'^2=\frac{2mn(m^2+n^2)}{m^2-n^2}\text{......(6).}$$

m and n must be selected so as to make (3) and (6) integral.

70. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

Give methods for decomposing numbers into squares, cubes, or biquadrates, and show that 61×200^3 is the sum of ten cube numbers, and that 844933 is the sum of eleven biquadrates in thirteen different ways. [From *The Mathematical Magazine*, Vol. II, No. 10.]

Comment by the PROPOSER.

Dr. Artemas Martin has written me that he knows of no method of separating a given number into squares, cubes and other powers, except *trial*.

I take the following numbers from *The Mathematical Magazine*.

$61\times200^3=1^3+2^3+4^3+6^3+15^3+46^3+60^3+270^3+500^3+700^3$.

$844933=27^4+20^4+16^4+15^4+12^4+10^4+8^4+7^4+3^4+2^4+1^4$,

$=25^4+24^4+16^4+15^4+8^4+6^4+5^4+4^4+3^4+2^4+1^4$,

$=25^4+24^4+15^4+14^4+12^3+10^4+7^4+4^4+3^4+2^4+1^4$,

$=25^4+22^4+18^4+15^4+14^4+11^4+10^4+6^4+3^4+2^4+1^4$,

$=25^4+21^4+18^4+17^4+14^4+12^4+10^4+6^4+5^4+4^4+1^4$,

$=25^4+21^4+18^4+16^4+15^4+12^4+10^4+9^4+6^4+3^4+2^4$,

$=25^4+21^4+18^4+15^4+14^4+13^4+12^4+10^4+8^4+7^4+2^4$,

$=25^4+20^4+19^4+18^4+14^4+10^4+9^4+7^4+6^4+4^4+3^4$,

$=25^4+20^4+19^4+16^4+14^4+13^4+11^4+10^4+9^4+4^4+2^4$,

$=24^4+23^4+18^4+15^4+14^4+12^4+10^4+9^4+6^4+5^4+3^4$,

$=24^4+21^4+20^4+16^4+15^4+13^4+10^4+7^4+6^4+4^4+1^4$,

$=24^4+21^4+20^4+16^4+15^4+12^4+11^4+8^4+7^4+5^4+2^4$,

$=24^4+21^4+19^4+18^4+15^4+12^4+10^4+6^4+5^4+3^4+2^4$.

MISCELLANEOUS.

64. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

How many bushels of wheat will a conical bin 8 feet in diameter at base and 12 feet high hold, if part of the bin is cut off by a plane parallel to the side and passing through the center of the base?

Solution by the PROPOSER.

Let ABC be the cone, DF the plane.

Let $BD=c$, $\tan DBC=\cot FDC=n$, $DC=R$.

$\therefore x^2+z^2=n^2(c-y)^2$ is the equation to the cone.

$x=ny$ is the equation to the plane.

V=volume of cone cut off by plane.

The limits of x are $ny=x_2$ and $n(c-y)=x_1$; of y, 0 and $\frac{1}{2}c$.

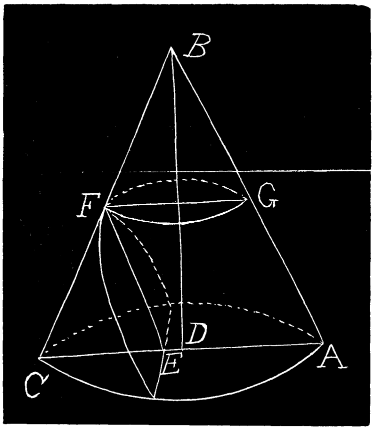
$$\therefore\ V=2\int_0^{\frac{1}{2}c}\int_{x_2}^{x_1}\sqrt{[n^2(c-y)^2-x^2]}\,dydx$$

$$=\int_0^{\frac{1}{2}c}\{\tfrac{1}{2}\pi n^2(c-y)^2-n^2(c-y)^2\sin^{-1}\left(\frac{y}{c-y}\right)$$

$$-ny\sqrt{[n^2(c-y)^2-n^2y^2]}\}dy$$

$$=\tfrac{1}{6}\pi n^2c^3-\tfrac{2}{9}n^2c^3=\tfrac{1}{18}n^2c^3(3\pi-4)$$

$$=\tfrac{1}{18}R^2c(3\pi-4).$$



But $c=12$, $R=4$.

$\therefore\ V=32\pi-\frac{128}{3}$.

Volume of cone$=\frac{1}{3}\pi R^2c=64\pi$.

$\therefore$ Required vol.$=64\pi-(32\pi-\frac{128}{3})=32\pi+\frac{128}{3}$

$=143.1978$ cubic feet,

$=115.07$ bushels.

Also solved by *P. S. BERG* and *C. C. CROSS*.

[NOTE.—In the figure, the point E should coincide with D. ED. F.]

65. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Show that the path of a projectile moving with a constant velocity is an inverted catenary of equal strength.

No solution has yet been received.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

102. Proposed by ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Iowa.

A's age is to B's as 2:3. 20 years from now their ages will be to each other as 4:5. What are their ages, respectively ?

103. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

Find proceeds of a note discounted at a bank for 10 years at 10%. What is the meaning of the result ?

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

ALGEBRA.

92. Proposed by ELMER SCHUYLER, High Bridge, N. J.

Given $x^2-yz=1$; $y^2-xz=2$; $z^2-xy=3$. Find x, y, and z.

93. **Proposed by CHARLES CARROLL CROSS, Libertytown, Md.**

Given $x^x+y^y=285$, and $y^x-x^y=14$, to find the values of x and y. [From *Bonnycastle's Algebra*, 1841.]

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

GEOMETRY.

108. **Proposed by NELSON L. RORAY, Bridgeton, N. J.**

ABC is a triangle. O_1, O_2, O_3 centers of escribed circles. Prove altitudes of triangle $O_1O_2O_3$ are concurrent at center of inscribed circle.

109. **Proposed by CHARLES CARROLL CROSS, Libertytown, Md.**

Two circles, radii in ratio 3:1, centers A and O_1 respectively, are drawn tangent externally to each other, and internally to a given circle O, and on the same diameter; O_2 and O_2' are drawn tangent internally to O and externally to A and O_1; O_3 and O_3' are drawn tangent internally to O and externally to A and O_2; O_3 and O_3' are drawn tangent internally to O and externally to A and O_2, A and O_2', respectively; and so on. Prove O_4, O, O_4'; O_5, A, O_5'; O_9, A, O_3' and O_{10}, O, O_2' are collinear. [The letters apply to the centers of the circles.]

110. **Proposed by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.**

If the three face angles of the vertical triedral angle of a tetraedron are right angles, and the lengths of the lateral edges are represented by a, b, and c, and of the altitude by p, then $1/p^2=1/a^2+1/b^2+1/c^2$. [*Chauvenet's Geometry.*]

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

CALCULUS.

83. **Proposed by J. SCHEFFER, A. M., Hagerstown, Md.**

From a given point, P, in the base AB of a triangle, to inscribe in the latter the minimum triangle, if its angle at P is given.

84. **Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.**

Find the equation of the curve upon which a given ellipse must roll in order that one of its foci may describe a straight line.

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

MECHANICS.

77. **Proposed by ELMER SCHUYLER, High Bridge, N. J.**

At what elevation must a shell be projected with a velocity of 400 feet that it may range 7500 feet on a plane which descends at an angle of 30°?

78. **Proposed by ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Iowa.**

A cone and a cylinder having equal heights and equal circular bases are filled with

water; if they have equal holes in the bases, respectively, how many times as long will it take the cylinder to empty as the cone?

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

DIOPHANTINE ANALYSIS.

76. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

It is required to find four positive numbers, such that if each be diminished by twice the cube of their sum the four remainders will be rational cubes.

77. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find (1) three consecutive numbers whose sum is a cube, and (2) three consecutive numbers the sum of whose cubes is a cube.

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

MISCELLANEOUS.

70. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

Prove $\tan^{-1}x=\frac{1}{2i}\left(\log\frac{x-i}{x+i}\right)$, and thence that $\pi=(2/i)\log(i)$.

71. Proposed by GUY B. COLLIER, 1901 Union, 27 Middle Section of South College, Schenectady, N. Y.

Find the locus of any point on the front sprocket of a bicycle during one revolution of the hind wheel (any gear may be assumed).

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

EDITORIALS.

Dr. Halsted reports that the Sylvester Fund has just been finished with about £900.

A copy each of the July number, Vol. II., and the July number, Vol. X., of the *Analyst* is wanted. Any one who can furnish these should write to the editor of the MONTHLY.

A number of important articles will appear in the MONTHLY during the next year. Dr. Miller will continue to furnish original matter on the all-embracing subject of Groups; Dr. E. O. Lovett will continue his series of articles on Lie's Transformation Groups; Dr. Roe has sent in a valuable article on Symmetric Functions, and Dr. Halsted has furnished a highly valuable account of the great Russian mathematician, Tchébychev. This biography will appear accompanied by an admirable picture of Tchébychev.

Nos. 6 and 11, of Vol. II. of the MONTHLY are exhausted. As we have yet a few orders for complete sets to fill, we shall be pleased to pay 40 cents apiece for copies of No. 6, Vol. II., and 30 cents apiece for copies of No. 11. There has been, of late, quite a demand for complete sets for College and University libraries. Whenever a sufficient number of orders are received to defray the expenses these numbers will be reprinted. Other numbers in other volumes of the MONTHLY are nearly exhausted. Subscribers desiring numbers to complete their files should order them at once.

In order to increase our subscription list for next year, we will allow each old subscriber who secures one new subscriber to remit us three dollars in full payment for one year's subscription to the MONTHLY for both.

BOOKS AND PERIODICALS.

Lectures on Elementary Mathematics. By Joseph Louis Lagrange. Being a course of lectures delivered at the École Normale, Paris, 1795. Translated from the French by Thomas J. McCormack. 8vo. Handsomely bound in Red Cloth. xvi+156 pages. Price, $1.00. Chicago: The Open Court Publishing Co.

In bringing out this valuable translation, the translator deserves great praise. These lectures which were delivered in 1795 and which are now found in the 7th volume of Lagrange's collected works have never before been published in separate form either in French or English, though a translation in German by Niedermüller appeared in 1880. These lectures are the source from which many of the best discussions of elementary mathematics have been drawn, besides containing much of great value and interest not yet incorporated in the text books. These lectures have received the indorsement of De Morgan, and Dühaing places them in the front rank of elementary expositions, as an exemplar of their kind.

The lectures discuss the theory of *Continued Fractions*, *Logarithms*, the *Operations of Arithmetic*, and fundamental principles generally. A brief *history of Algebra* is given and a full discussion of *equations of the third and fourth degree*, including the irreducible case. The two final lectures are devoted to the resolution of numerical equations, and to the usage of curves in the solution of problems.

Teachers of elementary mathematics, who desire to improve their methods of instruction, adding richness and vitality to the subject, should read this book. B. F. F.

A Manual of Experiments in Physics. Laboratory Instruction for College Classes. By Joseph S. Ames, Ph. D., Associate Professor of Physics in Johns Hopkins University, and William J. A. Bliss, Ph. D., Associate in Physics in Johns Hopkins University. 8vo. Cloth. 544 pages. New York: Harper & Brothers.

To my mind, this is by far the best laboratory manual that has appeared. B. F. F.

Elements of the Differential Calculus. By James McMahon, A. M. (Dublin), Assistant Professor of Mathematics in Cornell University, and Virgil Sny-

der, Ph. D. (Göttingen), Instructor in Mathematics in Cornell University. 8vo. Cloth. xiv+337 pages. Price, $2.00. New York, Cincinnati, and Chicago: American Book Co.

This book is the second of the Cornell Mathematical Series, of which series, Lucien Augustus Wait, Senior Professor of Mathematics in Cornell University, is editor.

The authors' apology for writing the book is that no other book has just the scope of this one, many of the works being too brief, and omitting rigorous proofs as being too difficult for the average student, while the more extensive treatises have too much for a student to master in the allotted time.

The authors have been very explicit in dealing with the fundamental principles of the science, and their efforts to reduce these principles to such clear and concise statements as to be readily comprehended by the student, will be greatly appreciated.

On page 59 we find the following method of differentiating $\log_a x$:

Let $y=\log_a x$. Then $y+\Delta y=\log_a(x+\Delta x)$.

$$\frac{\Delta y}{\Delta x}=\log_a\frac{(x+\Delta x)-\log_a x}{\Delta x}$$

$$=\frac{1}{\Delta x}\log_a\left(\frac{x+\Delta x}{x}\right)=\frac{1}{x}\cdot\frac{x}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)=\frac{1}{x}\log_a\left(1+\frac{\Delta x}{x}\right)^{x/\Delta x}.$$

. . . .

$$\frac{dy}{dx}=\frac{1}{x}\lim_{\Delta x\doteq 0}\left[\log_a\left(1+\frac{\Delta x}{x}\right)\right]^{x/\Delta x}=\ldots\ldots\frac{1}{x}\log_a e.$$

It is then stated in a foot-note that this method is too brief to be rigorous, it assuming that $x/\Delta x$ is a positive integer, which is equivalent to restricting Δx to approach zero in a particular way. A rigorous and general proof is given in the Appendix, page 316, in which the above restriction is avoided. However, while the method of proof there given is rigorous, we believe the following to be equally so:

Let $y=\log_a x$, $a>1$. Then $y\pm\Delta y=\log_a(x\pm\Delta x)$,

the upper signs corresponding and the lower signs corresponding.

$$\therefore\ \pm\Delta y=\log_a(x\pm\Delta x)-\log_a x=\log_a\left(1\pm\frac{\Delta x}{x}\right).$$

$$\frac{\Delta y}{\Delta x}=\pm\frac{1}{\Delta x}\log_a\left(1\pm\frac{\Delta x}{x}\right)=\pm\frac{1}{\Delta x}\cdot\frac{x}{x}\log_a\left(1\pm\frac{\Delta x}{x}\right)=\frac{1}{x}\log_a\left(1\pm\frac{\Delta x}{x}\right)^{\pm(x/\Delta x)}.$$

$$\therefore\frac{dy}{dx}=\frac{1}{x}\lim_{\Delta x\doteq 0}\log_a\left(1\pm\frac{\Delta x}{x}\right)^{\pm(x/\Delta x)}=\frac{1}{x}\lim_{\Delta x\doteq 0}\log_a\left[\left(1^{\pm(x/\Delta x)}+\left(\pm\frac{x}{\Delta x}\right)\left(\pm\frac{x}{\Delta x}\right)\right.\right.$$

$$+\frac{1}{2!}\left(\pm\frac{x}{\Delta x}\right)\left(\pm\frac{x}{\Delta x}-1\right)\left(\pm\frac{\Delta x}{x}\right)^2+\frac{1}{3!}\left(\pm\frac{x}{\Delta x}\right)\left(\pm\frac{x}{\Delta x}-1\right)\left(\pm\frac{x}{\Delta x}-2\right)\left(\pm\frac{\Delta x}{x}\right)^3$$

$$\left.+,\text{ etc.}\right]=\frac{1}{x}\lim_{\Delta x\doteq 0}\log_a\left[1+1+\frac{1}{2!}\left(1\mp\frac{\Delta x}{x}\right)+\frac{1}{3!}\left(1\mp\frac{\Delta x}{x}\right)\left(1\mp\frac{2\Delta x}{x}\right)+,\text{ etc.}\right]$$

$$=\frac{1}{x}\log_a\left[1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+,\text{ etc.}\right]=\frac{1}{x}\log_a e.$$

Δx is independent of x and can approach 0 in two ways only, viz., either from the left or from the right.

The same method holds when $a<1$, in which case $dy/dx=(1/x)\log_a(1/e)$.

The work is an excellent one and it together with the Integral Calculus of this series will make a most valuable course in this subject. We very heartily recommend this work to the consideration of teachers desiring something new and at the same time good.

B. F. F.

INDEX.

CALCULUS.

MECHANICS.

DIOPHANTINE ANALYSIS.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. DECEMBER, 1898. No. 12.

BIOGRAPHY.

TCHÉBYCHEV.

BY GEORGE BRUCE HALSTED.

PAFNUTI LVOVICH CHEBYSHEV in the notice which appeared in THE AMERICAN MATHEMATICAL MONTHLY, Vol. II., pp. 61-63, shortly after his death, was ranked as second only to Lobachévski among Russian mathematicians. In the memoir of Vasiliev which has just appeared the author says that if Russian mathematics may claim an honorable place in the modern science it is due to Lobachévski, to Ostrogradski, to Chebyshev.

The biography of Sylvester in the Proceedings of the Royal Society says of Sylvester's discourse on Conversion of Motion: "A synopsis only of the Royal Institution lecture was published. The manuscript of the lecture as actually delivered is in the possession of George Bruce Halsted, of the University of Texas. Extracts from it appear in the Amerlcan Journal 'Science', of April 16, 1897, from which it appears that it was characterized by that eloquence, force, and poetical imagination with which students of Sylvester are familiar." From this precious unpublished MS. of the great "king of thought" I give the following: "My friend, Professor Tchebicheff, of the University of St. Petersburg (one of the greatest mathematicians of this or any previous age, the only man ever able to cope with the refractory character and erratic flow of prime numbers and to confine the stream of their progression within algebraic limits, building up, if I

may so say, banks on either side which that stream, devious and irregular as are its windings, can never overflow), had long occupied himself with the theory of Parallel Motions. When he paid his first visit to me in England, at Lincoln's Inn Fields, about twenty years ago, his special object was to obtain sight of some of the old machines of Watt in order to ascertain if the parallel motions constructed by him followed strictly the description ordinarily given of them, or were adjusted in a manner to give more accurate results for a throw of the beam not indefinitely small, as the common theory virtually implies, but ranging within given limits.

I believe he found that the intuitive mechanical sagacity of Watt had anticipated his own mathematical deductions.

Since that time, among other researches of far greater importance, Tchebicheff has occupied himself with finding the best form of the 3-bar motion, of which Watt's parallel motion is only a special case, and arrived at results which give greater accuracy than Watt's at the same time that his arrangement is more compact, and, in some cases, more convenient of application. I may mention that in Tchebicheff's arrangement, the two radial bars cross instead of being parallel in the initial position; and that while in Watt's arrangement the parallel point (*i. e.* the part meant to move straight) describes a figure of 8, in Tchebicheff's system it describes a very flat but open oval curve.

As regards the problem of a perfect parallel motion, Tchebicheff had satisfied himself that the solution was impossible, and had elaborated what seemed to him *almost* a conclusive proof that such a machine was inconstructible in the nature of things,—almost but not quite; for under the very special hypothesis of two points usually distinct chancing to come into coincidence (if that could happen), he tells me that he foresaw that there was a bare possibility of his proof not holding water, and therefore with most praiseworthy caution, he held it back until he could succeed in patching up the supposed flaw. Still he made no secret of his inner conviction that the problem did not admit of solution; nor did he stand alone among mathematicians of the highest rank in entertaining that belief.

On his recent visit (the third I believe which he has paid) to England, in the course of last autumn, he again called upon me, and naturally, knowing how much the subject had occupied his thoughts, I inquired after his old love, and asked him how fared his proof of the impossibility of a Perfect Parallel Motion. 'Oh,' replied my eminent guest, 'it is done!' 'Done!' said I, offering him my hand, 'I may then congratulate you on having at last overcome the difficulty with which you have so long contended.' 'No,' said he, with a somewhat downcast air, 'it has been done in France and again in Russia.' 'Somebody else, then, has obtained a proof of the problem being impossible.' 'No; the problem has been solved, so that of course it cannot be demonstrated to be impossible.'

He then showed me Peaucellier's figure, which did indeed make me open my eyes, so simple—seeming to have so long eluded the gaze of the many sharp-sighted telescopic explorers watching to discover it flit over the wires! 'Now,

Dr. Tchebicheff,' said I, what do you think of the quadrature of the circle? Does not such a discovery shake our belief very much in what we call our intuitive convictions about things being possible or impossible?' 'My feeling,' was his answer, 'about the quadrature of the circle is this, that if the Emporer were to give me my choice to be shot at the expiration of a given time if I did not previously find out the quadrature, or else if I did not prove it to be impossible, I would elect to have to discover the quadrature.' [This was in 1874.]

He told me, too, how a pupil of his own, named Lipkin, a freshman at the University of St. Petersburg, had rediscovered Peaucellier's method in ignorance of what Peaucellier had done, and how at his own special recommendation, the Russian government had taken Lipkin in hand, and provided him with means to free himself from all anxiety for a livelihood, and to enable him to pursue his studies at the University with his mind at ease.

The air of Russia seems to no less favorable to mathematical acumen than to a genius for fable and song. Lobacheffsky, the first to mitigate the severity of the Euclidean code and to beat down the bars of a supposed adamantine necessity, was born (a Russian of Russians) in the government of Nijni Novgorod; Tchebicheff, the prince and conquerer of prime numbers, (who, rightly apprehending every fact as an actualized center, a ganglion of intellectual notions, seeks in realistic constructions and problems of every day life the source of his highest and subtlest mathematical inspirations), in the adjacent circumscription of Moscow, and our own Cayley, the central luminary of the mathematical firmament, was cradled amidst the snows of St. Petersburg.

I happened to receive a visit from M. Manuel Garcia immediately after Tchebicheff left me. I showed M. Garcia, who is a zealous geometer, the drawing of the cell and mounting of Peaucellier's instrument left by Tchebicheff. The next day to my surprise and gratification he brought me a working model of it in wood. When I took this instrument under my cloak and showed it after dinner to the Philosophical Club of the Royal Society it drew forth the most lively expressions of admiration from such men as Wheatstone. Tyndall, Sir C. Lyell, Dr. Carpenter, Mr. Busk, Sir B. Brodie, Mr. Playfair, and others. Soon after I exhibited the same model in the hall of the Athenaeum Club to my brilliant friend Sir William Thompson of Glasgow, who nursed it as if it had been his own child, and when a motion was made to relieve him of it replied: 'No! I have not had nearly enough of it—it is the most beautiful thing I have ever seen in my life.'

Had Peaucellier gone boldly to the Institute with his discovery, I feel fully persuaded from my personal knowledge of the men who adorn its lists, that he would not have had to wait nine years or to have depended on my accidental conference with Tchebicheff for his claims to have met with their first suitable recognition."

In a note, Sylvester says Tchebicheff told him he had succeeded in proving the non-existence of a five-bar linkwork capable of producing a perfect parallel motion. Peaucellier's required seven. But again the great Russian was

mistaken; for fired and inspired by this very lecture, Harry Hart went home and discovered the five-bar perfect parallel motion now known by his name, where for Peaucellier's cell of six bars is substituted Hart's contraparallelogram of four.

ON A METHOD TO CONSTRUCT INTRANSITIVE SUBSTITUTION GROUPS.

By DR. G. A. MILLER.

An intransitive group contains two or more transitive constituents and may be constructed by establishing some isomorphism between these constituent groups. If it contains more than two transitive constituents we may combine them in any way into two constituents and construct the group by establishing some isomorphism between these constituents, where at least one of them is an intransitive group. For example, we can find all the possible groups of degree 11 whose transitive constituents are of degrees 4, 4, 3 by establishing isomorphisms between the intransitive groups of degree 8 which contain two transitive constituents of degree 4 and the transitive groups of degree three. We can also find all these groups by establishing isomorphisms between the intransitive groups of degree 7 whose systems of intransitivity are 4, 3 and the transitive groups of degree 4.

Sometimes we can observe some important group properties from the notation by which an intransive group is represented. For instance, if the group is not simply isomorphic to some one of its transitive constituents it must contain at least two self-conjugate subgroups, differing from identity, that do not have any common operator besides identity. Conversely, if a group contains two such self-conjugate subgroups it may be represented as an intransitive group which is not simply isomorphic to any one of its transitive constituents.

While the method of establishing some isomorphism between two constituent groups seems to be the best general method to construct intransitive groups yet it is sometimes desirable to employ others. To illustrate one of these we shall employ it to find all the possible intransitive groups of degree 10 that contain five systems of intransitivity.

The average number of elements in all the substitutions of such a group is $10-5=5$* and a positive substitution must be of degree 4 or 8, while a negative substitution must be of degree 2, 6, or 10. There is evidently only one such group of order 2. If a group of order 4 contains only positive substitutions it must contain x substitutions of degree 4 and y of degree 8, where $x+y=3$. Hence

*Frobenius, *Crelle*, vol. 101, p. 287; cf. Miller, *Bulletin of the American Mathematical Society*, vol. 2, 1895, p. 75.

$$4x+8(3-x)=4.5\,; \quad x=1, \quad y=2.$$

Since the substitution of degree 4 and one of the substitutions of degree 8 must generate the required group of degree 10 they can have only one transposition in common. Hence there is only one positive group of order 4.

If a group of order 4 contains negative substitutions it must involve one positive substitution besides identity. If this is of degree 4 the two negative substitutions must involve $20-4=16$ elements. Hence one of them must be of degree 6 and the other of degree 10. Since the one of degree 6 and that of degree 4 must generate the entire group there is only one such group. If the positive substitution is of degree 8, the two negative substitutions must involve $20-8=12$ elements. There is clearly one group whose negative substitutions are composed of one transposition and one substitution of degree 10, and one whose negative substitutions contain three transpositions. Hence there are four groups of order 4,—one of these contains only positive substitutions while the other three contain negative substitutions.

If a group of order 8 contains only positive substitutions it must contain x substitutions of degree 4 and y of degree 8, where $x+y=7$. Hence

$$4x+8(7-y)=8.5\,; \quad x=4.$$

The substitutions of degree 4 must therefore generate the entire group. Since at least two of them must have a common transposition such a group must contain a subgroup of order 4 and degree 6. There is only one such group because the remaining generating substitution of degree four must involve the 4 elements that are not found in the given subgroup of order 4.

We proceed to consider the groups of order 8 that include negative substitutions. The positive subgroup of order 4 must be of degree 6, 8, or 10. If it is of degree 6 its substitutions must involve 12 elements. The four negative substitutions of the group must therefore involve $40-12=28$ elements. Since there must be one substitution of degree 10 there can be only one such group. If the given subgroup is of degree 8 the four negative substitutions of the group must include $40-16=24$ elements. There is clearly one group that contains a single transposition and another that contains four substitutions of degree 6.

Finally, if the given positive subgroup is of degree 10 the four negative substitutions must involve $40-20=20$ elements. Hence at least one of these substitutions must be a transposition. If this transposition is found in the substitution of degree 4 the group will contain two transpositions ; if it is not contained in this substitution the group will contain only one transposition. We have now considered all the possible cases and found that there are six groups of order 8,—one of these is positive while the other five contain negative substitutions.

The largest possible group of degree 10 that contains five systems of intransitivity is of order $2^5=32$. There is evidently only one group of this order. From this it follows directly that there is only one positive group of order 16 that contains the given systems of intransitivity. The three groups of this order that

contain negative substitutions may be found in exactly the same manner as those of order 8. Hence there are just 16 groups of degree 10 that contain five systems of intransitivity,—1 of order 2, 4 of order 4, 6 of order 8, 4 of order 16, and 1 of order 32. All those that have the same order represent the same abstract group.

Cornell University, November, 1898.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from May Number.]

PROPOSITION XXXVI. *If any straight XF* (Fig. 44) *makes an acute angle with any ordinate LF [of the equisdistantial], the point X does not fall without the cavity of the curve, unless previously XF has cut the curve in some point O.*

PROOF. It is certain that the point X may be assumed in XF so near to the point F, that the join LX previously cuts the curve in some point S: otherwise XF either does not fall wholly without the cavity of the curve, and so we have our assertion; or so it does not make with FL an acute angle, rather it would be to suppose that XL comes together with LF in one same straight.

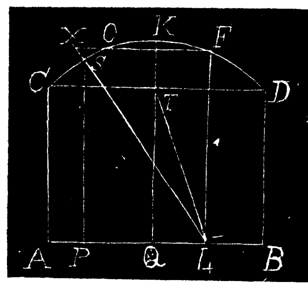


Fig. 44.

Accordingly from the point S let fall to the base AB the perpendicular SP. This will be (from P. 34) equal to LF.

But SP is (from Eu. I, 18) less than LS.

Therefore also LF is less than LS, and therefore much less than LX. Hence in triangle LXF the angle at the point X will be acute, because less (from Eu. I, 18) than the angle LFX supposed acute.

Now let fall to FX the perpendicular LT. This falls (because of Eu. I, 17) toward the parts of each acute angle. Wherefore the point T will lie between the points X and F. Then from the point T let fall to the base AB the perpendicular TQ.

LF (because of the right angle at T) will be greater than LT, and this (because of the right angle at Q) will be greater than QT. Therefore LF will be far greater than QT. But hence, if in QT produced QK is taken equal to LF, the point K (from P. 34) will pertain to the present curve, and therefore the point T falls within the cavity of this curve.

Therefore the straight FT, which cuts two straights QK and LT in T, can

not attain to cutting LS produced in the point X, situated without the cavity of the present curve, unless previously the prolonged FT' cuts in some point O the portion of this curve between the points S and K.

Hoc antem erat demonstrandum.

COROLLARY. And hence flows manifestly, that between the tangent of this curve and the curve itself cannot be placed any straight [ray], whether on one or the other side of the tangent, which falls wholly without the cavity of the curve; since a straight [ray] so located must make (from the preceding) an acute angle with the perpendicular let fall from the point of contact to the opposite base.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

102. Proposed by ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Iowa.

A's age is to B's as 2:3. 20 years from now their ages will be to each other as 4:5. What are their ages, respectively?

I. Solution by J. OWEN MAHONEY, B. E., M. Sc., Master of Mathematics and Science, Carthage Graded and High School, Carthage, Texas.

It is easily seen that A's age : A's age + 20 years :: 2 : 4,

or A's age : 20 years :: 2 : 2.

$\therefore$ A's age = 20 years. But B's age = $\frac{3}{2}$ of A's = 30 years.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

In the first instance, A's age = $\frac{2}{3}$ B's age.

In the second instance, A's age = $\frac{4}{5}$ B's age.

$\therefore$ $\frac{2}{3}$ B's age + 20 years = $\frac{4}{5}$(B's age + 20 years).

$\therefore$ $\frac{2}{15}$ B's age = 4 years.

$\therefore$ B's age = 30 years, A's age = 20 years.

An algebraic solution was furnished by Charles C. Cross.

Solutions of problems 98 and 99 were received from J. K. Ellwood, and solutions of problems 100 and 101 were received from J. Scheffer too late for credit in last issue.

ALGEBRA.

89. Proposed by G. A. MILLER, Ph. D., Instructor in Mathematics, Cornell University, Ithaca, New York.

Solve by quadratics, $x^2 + y = 7 \ldots\ldots (1)$,

$x + y^2 = 11 \ldots\ldots (2)$.

I. Solutions by W. F. BRADBURY, A. M., Head Master Cambridge Latin School, Cambridge, Mass.

The late Professor Quimby, of Dartmouth College, pronounced equations of this form insoluble except by Descarte's formula for bi-quadratics.

Various ways of getting these values of x and y in these particular equations I have seen.

(a) If x and y are integral and positive, from (1) we know x^2 must be 1 or 4; and from (2) we see that x^2 cannot be 1. Hence $x^2=4$, and $y=2$ (not ± 2). Hence $y=3$.

(b) Adding (1) and (2), multiplied by 4, and adding 2 to both members, we have $4x^2+4x+1+4y^2+4y+1=74$, or $(2x+1)^2+(2y+1)^2=74$.

Now if x and y are both integral then 74 is the sum of two perfect squares; the only perfect squares whose sum is 74 are 25 and 49, and as $y>x$ it follows that $(2x+1)^2=25$, and $(2y+1)^2=49$, or $x=2$, and $y=3$. The answers obtained from cålling $2x+1=-5$ and $2y+1=-7$ do not prove.

(c) Multiply (1) by x, then $xy^2+x^2=11x$..... (3). From (3) subtract (2), $xy^2-y=11x-7$......(4). To (4) add twice (1), $xy^2+2y^2-y+2x=11x+15$ ......(5), or $(x+2)y^2-y=9x+15$, or $y=3$, or $(-3x-5)/(x+2)$, and $x=2$, or $x^2=7+(3x+5)/(x+2)$.

This $x^2=7+(3x+5)/(x+2)$ is a cubic equation which gives three other values for x.

II. Solutions by M. E. GRABER, Student in Heidelberg University, Tiffin, O.; W. F. BRADBURY, A. M., Cambridge, Mass.; and P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

Equating values of x^2 from (1) and (2), $7-y=(11-y^2)^2$......(3), $y^4-22y^2+y+114=0$..... (4).

Factoring $(y-3)(y^3-3y^2-13y+38)=0$.

$\therefore y-3=0$ and $y=3$, and $x=2$. The three other values of y can be found by reducing $y^3+3y^2-13y+38=0$. GRABER, BRADBURY, BERG.

Let $z=y+1$, and substitute for y in the latter disjuntive equation $y^3+3y^2-13y+38=0$, and we get $z^3-16z-23=0$. Assume $z=\omega\sqrt[3]{(r_1)}+\omega^2\sqrt[3]{r_2}$.

$\therefore r_1+r_2=23$, and $\sqrt[3]{(r_1r_2)}=\frac{16}{3}$. $\therefore r_1r_2=\frac{4096}{27}$.

$\therefore r=\frac{1}{2}[23\pm\sqrt{(23^2-\frac{16384}{27})}]=\frac{1}{2}[23\pm\frac{1}{9}i\sqrt{(6303)}]$.

$\therefore y=3$ or $-1+\omega\sqrt[3]{[11\frac{1}{2}+\frac{1}{18}i\sqrt{(6303)}]}+\omega^2\sqrt[3]{[11\frac{1}{2}-\frac{1}{18}i\sqrt{(6303)}]}$ and $x=2$ or $\sqrt{\{11-[-1+\omega\sqrt[3]{(11\frac{1}{2}+\frac{1}{18}i\sqrt{6303})}+\omega^2\sqrt[3]{(11\frac{1}{2}-\frac{1}{18}i\sqrt{6303})}]\}}$.

P. S. BERG.

III. Solution by A. H. BELL, Hillsboro, Ill., and W. F. BRADBURY, A. M. Cambridge, Mass.

Write (1) $x^2-4=3-y$......(1).

Write (2) $9-y^2=x-2$......(2).

$(1)\times(2)$, etc., $(x^2-4)(9-y^2)-(x-2)(3-y)=0$......(3).

$\therefore$ We have factors $x-2=0$, $x=2$; $3-y=0$, $y=3$.

The other values of x and y can be easily found. None of these methods can be applied to $x^2+y=a$ and $x+y^2=b$.

IV. Solution by SYLVESTER ROBINS, North Branch Depot, N. J.

Every quantity which is twice a square, as 2, 8, 18, 32, 50, 72, 98, 128,

162, 200, etc., is the sum of two consecutive numbers plus their squares. This is readily seen from $a^2+(a+1)^2+a+(a+1)=2a^2+4a+2=2(a+1)^2$.

Consequently in $x^2+y+x+y^2=7+11=18$, the greater of the two unknowns is $=\sqrt{[\frac{1}{2}(18)]}=3$, and the less is a unit smaller $=2$.

V. **Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.**

(a) Adding (1) and (2), $x^2+x+y^2+y=18$, or $x^2+x+\frac{1}{4}+y^2+y+\frac{1}{4}=18+\frac{1}{4}+\frac{1}{4}$. $(x+\frac{1}{2})^2+(y+\frac{1}{2})^2=\frac{74}{4}=\frac{25}{4}+\frac{49}{4}$.

Whence $x+\frac{1}{2}=\frac{5}{2}$ or $\frac{7}{2}$, $y+\frac{1}{2}=\frac{7}{2}$ or $\frac{5}{2}$, or $x=2$, $y=3$, the other answer not being applicable.

(b) From (1), $y-3=4-x^2$; from (2), $y^2-9=2-x$, or $y-3=\dfrac{2-x}{y+3}$.

Whence $\dfrac{2-x}{y+3}=4-x^2$, or $x^2-\dfrac{x}{y+3}=4-\dfrac{2}{y+3}$, or $x^2-\dfrac{x}{y+3}+\dfrac{1}{4(y+3)^2}=4-\dfrac{2}{y+3}+\dfrac{1}{4(y+3)^2}$. $x-\dfrac{1}{2(y+3)}=2-\dfrac{1}{2(x+3)}$,

whence $x=2$, and hence $y=3$.

(c) From (1), $y=7-2$.

Substituting in (2), $x+49-14x^2+x^4=11$, or $x^4-14x^2+x+38=0$.

Factoring $(x-2)(x^3+2x^2-10x-19)=0$,

whence $x=2$ for one root.

The cubic $x^3+2x^2-10x-19$ have 3 real roots, two negative and one positive. The positive root is between 3 and 4; the negative roots between -1 and -2, and -3 and -4.

These are found by Horner's method to be as follows:

$$x_1=3.13+,\quad x_2=-1.74+,\quad x_3=-3.27+.$$

From these the corresponding value of y can be found.

For a different solution see *Fisher and Schwatt's Algebra*, page 576.

VI. **Solution by G. H. RICHARDS, Hillsboro, Ill.**

$(2)-(1)$ $y^2-x^2=4+y-x=1(4+y-x)$, $(y+x)(y-x)=1(4+y-x)$ ....(3).

$\therefore$ $y-x=1$, $y+x=4+y-x$ ....(4, 5).

$(4)+(5)$ gives $2y=5+y-x$. $y=5-x$.

$\therefore$ $x^2-x=2$. $\therefore$ $x=2$, $y=3$.

VII. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

Let $t=3-y$. This value of $3-y$ in (1) and (2) gives

$$x^2-4=t\ldots.(3),\quad x-2=t(3+y)\ldots.(4).$$

$\therefore$ $x^2-4=\dfrac{x-2}{3+y}$. $\therefore$ $x^2-\dfrac{x}{3+y}=4-\dfrac{2}{3+y}$.

$$\therefore x^2-\frac{x}{3+y}+\frac{1}{4(3+y)^2}=4-\frac{2}{3+y}+\frac{1}{4(3+y)^2}.$$

$$\therefore x-\frac{1}{2(3+y)}=2-\frac{1}{2(3+y)}.$$

$\therefore x=2,\ y=3.$

From (1), $y=7-x^2$. This in (2) gives $x^4-14x^2+x+38=0$.

$\therefore x=2 \qquad y=3$

$x=3.13132 \qquad y=-2.80516$

$x=-3.28316 \qquad y=-3.77914$

$x=-1.84816 \qquad y=3.5843$

$x^4-14x^2+x+38=(x^2-5\frac{13}{99}x+6\frac{26}{99})(x^2+5\frac{13}{99}x+6\frac{6664}{9801})$ approximately.

$\therefore x^2-5\frac{13}{99}x+6\frac{26}{99}=0. \quad x^2+5\frac{13}{99}x+6\frac{6664}{9801}=0.$

$\therefore x=2 \qquad y=3$

$x=3\frac{13}{99}=3.1313 \qquad y=-2.8052$

$x=-3.2832 \qquad y=-3.7792$

$x=-1.8481 \qquad y=3.5843$

Also solved by *W. H. DRANE.*

VIII. Solution by J. M. BOORMAN, Consultative Mechanician and Counselor at Law, Woodmere, Long Island, N. Y.

"Empirical" and "Practical" (note page 174, problem 83) each find all the roots by quadratics. "Empirical" (page 173) is here ($fx=x$).

$\therefore \Sigma^4-2(11\pm\sqrt{114})\Sigma^2=\frac{1}{4}.(1/\sqrt{114})$, $\sqrt{114}=$........10.677078,252031.

Complete the square by squaring $(11+\sqrt{114})$, which squared

$=125+22\sqrt{114}=469.895721,544682$

and adding to both sides

We have $\Sigma^4-2[11+\sqrt{(11^2-7)}]\Sigma^2+469.8957$, etc., $=$......469.919136,189971

since $\frac{1}{4}.(1/\sqrt{114})=$...0.023414,645289

Extracting square roots, $\Sigma^2=$......21.677078,252031+21.677618,323745 $=43.354696,575776$ and $\Sigma=\pm6.584428,340849$....(3).

True $\Sigma=6.584428,340330,491744,944338,239380,16$, etc....($A$).

Error$=0.000000,000518+$.

To save labor, as (2) transposes to $y^2-9=2-x$, therefore $y-3=[2/(y+3)]-[x(y+3)]$. $\therefore y=3+[2(y+3)]-[x(y+3)]$....($B$). By ($B$) in (1) $x^2-[1/(y+3)]x+\square$, $=7-3-[1/(y+3)]2+\square$, $=4-[1(y+3)]2+\square$, ...(C). Extracting the square root and solving for x, $x=\frac{1}{2}\{[1(y+3)]\pm_1[1/(y+3)]\}\mp(-2)$....($D$)!

$\therefore x=[1/(y+3)]-2$....(E); $x_1=2$....(F). By (F) in (1) $y_1=3$, $\therefore$ 3 from (A) is $y_a=3.584428,340330,491744,944338,239380,161894,6\pm$

and $x_a=-1.848126,526964,403553,538300,209036,266178,2\mp$

$x_1+x_a=S=(1/\Sigma)=0.151873,47+$....(4) and $(4)-2=-1.848126,5+$.

(E) in (1) gives $y^2+[1(y+3)]=13$, whence $y^3+3y^2-13y=38$....(G) has the three non-obvious roots. $\therefore x^3+2x^2-10x=19$....($H$) by parity. But y_a and x_a are known by Σ and S. Therefore easily by quadratics:

$x = 3.131312,518250,572965,804300,733409,211483,7\pm$

$y = -2.805118,086952,744853,053572,398087,397376,0\mp$

$x_{11} = -3.283185,991286,169412,266000,514372,745305,5\pm$

$y_{11} = -3,779310,253377,746891,890765,841292,764578,6\mp$

$^*S^3 \mp_1 2xS^2 - 2yS = \mp f \ldots (J)$ by $y = \mp_1 Sx + \frac{1}{2}(S^2 \pm_1 \Sigma) \ldots (K)$ in (I).

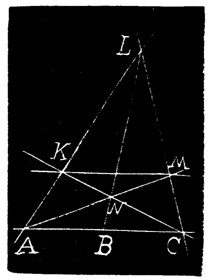
$\therefore\ x^2 \mp_1 Sx = 7 - \frac{1}{2}(S^2 \pm_1 \Sigma)$ give by quadratics all x's and y's with no further aid.

GEOMETRY.

102. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Ohne Benutzung des Cirkels eine Strecke AC zu halbiren, wenn eine Parallele der Geraden AC gegeben ist. [Reye's Geometrie der Lage, Part I, page 191].

I. Solution by C. HORNUNG, Professor of Mathematics, Heidelberg University, Tiffin, Ohio.



Wir ziehen durch irgend einen Punkt L auserhalb der gegebenen Parallele die Linien AL und CL welche die Parallele in resp. K und M schneiden. Dann ziehen wir die Linien AM und CK die sich im Punkte N schneiden. So bestimmt die Diagonale LN des Vierecks $KLMN$ den gesuchten Halbirungspunkt B der Strecke AC. Denn der vierte von B harmonisch getrennte Punkt, in welchem die zweite Diagonale KM des Vierecks die Gerade ABC schneidet, liegt uendlich fern.

II. Solution by the PROPOSER.

GIVEN the line AC and the line KM parallel to it,

TO BISECT AC without using the circle.

CONSTRUCTION. 1. Through A draw any two lines intersecting the parallel line in the points K and M.

2. Draw the lines CK and CM, CK intersecting AM in N and CM intersecting AK in L.

3. Draw the diagonal, LN, of the 4-side, $LMNK$, and its intersection B with AC will be the required point of bisection.

The points A, B, C, and the point at infinity, *i. e.* the intersection of AC and MK, constitute a harmonic range of points and the point at infinity is harmonically separated from B by the points A and C.

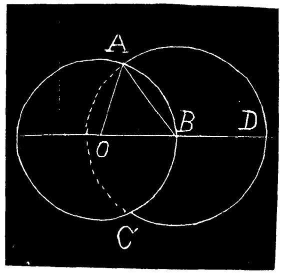
This problem was also solved by *G. B. M. ZERR, C. C. CROSS, J. SCHEFFER, A. F. KOVARIK, M. A. GRUBER, NELSON L. RORAY, ELMER SCHUYLER*, and *JOHN MANIE*.

103. Proposed by FREMONT CRANE, Sand Coulee, Mont.

A horse is tethered with a rope which is attached to a stake B on the edge of a circular pond containing one acre. How long must the rope be to allow the horse to graze over one acre ? [From *Home Study Magazine*, problem 249].

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; WALTER H. DRANE, Graduate Student, Harvard University, and M. A. GRABER, Tiffin, Ohio.

Let B be the point where the horse is tethered, A and C the points on the edge of the given pond, to which the horse can graze, O the center of the pond. Let $AO=r=4\sqrt{(10/\pi)}$, the radius of the pond, and $\angle ABO=\theta$.



$\therefore$ AB=length required$=2r\cos\theta$. Also the area common to the pond and the circle upon which the horse can graze$=r^2(\pi+2\theta\cos2\theta-\sin2\theta)$.

$\therefore$ The area upon which the horse can graze outside the pond is $4\pi r^2\cos^2\theta-r^2(\pi+2\theta\cos2\theta-\sin2\theta)=\pi r^2$. $\therefore$ $2\theta-\tan2\theta=2\pi$.

$\therefore$ $\theta=51°\ 16'\ 24''$. $\therefore$ $AC=2r\cos\theta=8.92926$ rods.

II. Solution by A. H. BELL, Hillsboro, Ill.

Let $10a=1$ acre, be the area of the pond, $10b$ the area grazed over, $R=\sqrt{(10a/\pi)}$ be its radius, and $AB=r=2R\sin\theta$ be the length of rope, where $2\theta=\angle AOB$. Then $r^2=4R^2\sin^2\theta=20a(1-\cos2\theta)/\pi$......(1).

2 sector $ABD-2$ segment $AB=10b$, or $r^2(\pi+2\theta)-2R^2(2\theta-\sin2\theta)=20b$. ..(2).

Substituting (1) in (2) and letting $\cos2\theta=x$ and $20b\pi/20a=c$, we get $2\theta x=\pi-c-\pi x+\sqrt{(1-x^2)}$, or

$$2\theta x=\tfrac{1}{2}\pi-c-\pi x+1-\tfrac{1}{2}x^2-x^4/2.4-3x^6/2.4.6-\text{ etc}....(3).$$

By trigonometry,

$$2\theta x=(\tfrac{1}{2}\pi)x-x^2-x^4/1.2.3-3^2x^6/1.2.3.4.5.6-\text{ etc}....(4).$$

(4)—(3) and transposing,

$$(\pi-c)+1=y=(3\pi/2)x-\tfrac{1}{2}x^2-x^4/2.3.4-x^6/4^2.5-\text{ etc}....(5).$$

Assume $\cos2\theta=x=ay+by^2+cy^3+dy^4+$ etc....(6).

Substituting (6) in (5) and equating the coefficients of like powers of y, we have $a=2/3\pi$, $b=4/27\pi^3$, $c=16/243\pi^5$, $d=(80+12\pi^2)/(2187\pi^7)$, etc.

Hence, $\cos2\theta=(2y/3\pi)+(4y^2/27\pi^3)+(16y^3/243\pi^5)+[(80+12\pi^2)y^4]/(2187\pi^7\pi)$ +etc. But $c=\pi$, and $y=1$.

$\cos2\theta=.212207+.004778+.000215+.000030+....=.21723$.

Hence $2\theta=77°\ 27'\ 12'$.

From (1) $r=2R\sin\theta=8.92$ rods.

III. Solution by CHARLES C. CROSS, Libertytown, Md.

Let a=length of rope, b=diameter of pond, and A=area over which the horse can graze. Then $(a^2/6b)(4a+3\pi b)=A$. [See Vol. IV, No. 1, of MONTHLY].

But by the problem $b=2\sqrt{(A/\pi)}$.

Substituting and reducing, we have,

$$a^3+\tfrac{3}{2}\sqrt{(\pi A)}a^2-3\sqrt{(A^3/\pi)}=0.$$

Restoring numbers, $a^3+33.62a^2-3425.514=0$,

whence $a=8.933+$ rods.

104. Proposed by SAMUEL E. HARWOOD, M. A., Professor of Mathematics, Southern Illinois State Normal University, Carbondale, Ill.

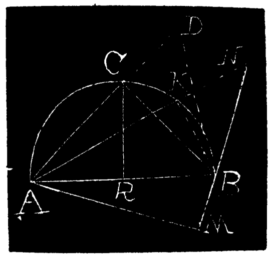
To find a point in a semi-circumference such that the sum of its distances from the extremities of the diameter shall be a maximum. [From *Wentworth's Plane Geometry*, Ex. 387.]

I. Solution by J. F. COWAN, Carbondale, Ill.

Through C, the middle point of the semi-circumference, draw AD, making $CD=CA=CB$. Then BD is perpendicular to AB. Ex. 46, Wentworth.

Through K, any other point in the semi-circumference, draw AN, making $KN=KB$.

Draw NB, producing it to meet AM at right angles.



$\angle ACB=\angle CBD+\angle CDB$.

Exterior angle of triangle BCD.

$\angle AKB=\angle KNB+\angle KBN$.

Exterior angle of triangle KBN.

$\angle ACB=\angle AKB$.

Right triangles.

Triangles CDB and KNB are isosceles.

By construction.

$\therefore$ $\angle CDB=\angle KNB$. $\therefore$ $\triangle ABD$ and $\triangle AMN$ are similar.

$AM<AB$. Base<hypotenuse of triangle.

$\therefore$ $AN<AD$. Homologous lines of similar triangles.

$\therefore$ $AK+KB<AC+CD$. C is the point. Q. E. F.

Mr. Cowan is a student of Southern Illinois State Normal University, and the above demonstration was sent to us by Prof. S. E. Harwood of the Department of Mathematics of that University.

A similar demonstration was given by *P. S. BERG*, *J. M. COLAW*, *H. F. STRATTON*, *WALTER H. DRANE*, *ALOIS F. KOVARIK*, and *ELMER SCHUYLER*.

II. Solution by J. O. MAHONEY, B.E., M. Sc., Master and Instructor in Mathematics and Science, Carthage Graded and High School, Carthage, Tex.; J. SCHEFFER, A. M., Hagerstown, Md.; CHARLES C. CROSS, Libertytown, Md.; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

From the center R of the semi-circle ACB draw RC perpendicular to AB, then C is the point required.

Proof: ACB is the maximum inscribed triangle.

AKB is any other inscribed triangle.

$AC^2+CB^2=AK^2+KB^2$.

$2AC.CB>2AK.KB$.

$AC^2+CB^2+2AC.CB>AK^2+KB^2+2AK.KB$, or $AC+CB>AK+KB$.

III. Solution by F. W. BRADBURY, A. M., Head Master, Cambridge Latin School, Cambridge, Mass.

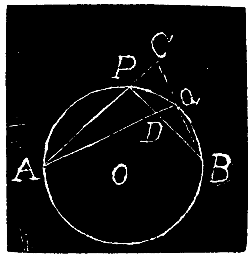
Let C be the middle point of the semi-circumference ACB, whose diameter is AB. Draw AC, BC; and CO perpendicular to AB. Then $AC+BC$ is a maximum. $AC>AO$, its projection on AB.

As the point C moves along the arc ACB, CO becomes less, and the line

AC becomes nearer equal to its projection on AB. Just so BC, when C is at A (or C) $AC+BC$=the sum of their projections on AB, or $AO+OB$.

That is, as C moves from the middle of the arc ACB, $AC+BC$ becomes less. $\therefore$ $AB+BC$ is a maximum when C is at the middle of the arc ACB.

IV. **Solution by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio.**



The required point is the mid-point of the semi-circumference.

Proof. Let Q be any other point in the semi-circumference than point P.

Produce AP and BQ until they meet, as at C.

Now $\triangle APD = \triangle BPC$, each being a right triangle, and having a leg and an acute angle of the one=to a leg and a homologous acute acute of the other.

$\therefore PC=PD$.

But $AC>AQ$, and $DB>QB$. Hence $AP+PD+DB>AQ+QB$, or $AP+PB>AQ+QB$. Q. E. D.

Other solutions were furnished by *NELSON L. RORAY, CHARLES C. CROSS, M. A. GRABER, J. M. COLAW*, and *COOPER D. SCHMITT*. Professors Cross, Colaw, and Yanney each furnished two solutions.

CALCULUS.

78. **Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.**

Investigate value of $\left(\frac{\tan x}{x}\right)^{1/x^n}$ where x is 0 and n has consecutive values 1, 2, 3, 4, ... Is there any law governing the different results? When $n=1$, result is 1; when $n=2$, result is $e^{\frac{1}{3}}$; $n=3$, gives ∞, etc.

II. **Solution by the PROPOSER.**

Let $y=\left(\frac{\tan x}{x}\right)^{1/x}$, making $n=1$, in the given expression.

Then $\log y=\frac{\log(\tan x/x)}{x}$ which is of the form 0/0.

$$\text{By the calculus this}=\frac{\dfrac{x\sec^2x-\tan x}{x^2}}{\dfrac{\tan x}{x}}=\left.\frac{x\sec^2x-\tan x}{x^2}\right]_{x=0}\text{ since }\left.\frac{\tan x}{x}\right]_{x=0}=1.$$

The numerator of this expression is the fundamental form which will occur in all the different values given to n.

Continuing the evaluating process we have

$$\left.\frac{x\sec^2x-\tan x}{x^2}\right]_{x\doteq 0}=\left.\frac{2x\sec^2x\tan x}{2x}\right]_{x\doteq 0}=\sec^2x\tan x\Big]_{x\doteq 0}=0.$$

Hence $y=e^0=1$.
Let $n=2$, and we have as before,

$$\left.\frac{x\sec^2x-\tan x}{2x^3}\right]_{x\doteq 0}=\left.\frac{\sec^2x\tan x}{3x}\right]_{x\doteq 0}=\left.\frac{\sec^4x+2\sec^2x\tan^2x}{3}\right]_{x\doteq 0}=\tfrac{1}{3}.$$

Hence $y=e^{\frac{1}{3}}$.
Let $n=3$, and we have

$$\left.\frac{x\sec^2x-\tan x}{3x^4}\right]_{x\doteq 0}=\left.\frac{\sec^2x\tan x}{6x^2}\right]_{x\doteq 0}=\left.\frac{\sec^4x+2\sec^2x\tan^2x}{12x}\right]_{x\doteq 0}=\tfrac{1}{0}=\infty.$$

Then $y=e^\infty=\infty$.
Let $n=4$, 5, 6...... and we have

$$\left.\frac{x\sec^2x-\tan x}{4x^5},\quad \frac{x\sec^2x-\tan x}{5x^6},\quad \dots\dots \frac{x\sec^2x-\tan x}{nx^{n+1}}\right]_{x\doteq 0}.$$

The numerator will remain as in the former cases, but the denominator will have increasing powers of x. Hence the result will always be ∞.

III. Note by Dr. E. D. ROE, Jr., Norwood, Mass.

In connection with this problem attention might perhaps be called to the fact the indeterminate form 1^∞ may often be more advantageously dealt with by referring it directly to the foundation of the form, viz, to

$$\operatorname*{Lim.}_{\varepsilon\doteq 0}\left(1+\varepsilon\right)^{1/\varepsilon}=e,$$

where ε is any expression whatever that $\doteq 0$, than by the usual method given in the books. Thus in the given example,

$$\frac{\tan x}{x}=1+\frac{x^2}{3}\left(1+\varepsilon\right),$$

where ε is a convergent series and therefore $\doteq 0$, as $x\doteq 0$.

$$\left(\frac{\tan x}{x}\right)^{1/x^n}=\left\{\left[1+\frac{x^2}{3}\left(1+\varepsilon\right)\right]^{\frac{3}{x^2(1+\varepsilon)}}\right\}^{\frac{1+\varepsilon}{3x^{n-2}}},$$

within the limits of convergence and certainly for small values of x. Therefore

$$\underset{x \doteq 0}{\text{Lim.}}\left(\frac{\tan x}{x}\right)^{1/x^n} = \underset{x \doteq 0}{\text{Lim.}}\left\{\left[1+\frac{x^2}{3}\left(1+\varepsilon\right)\right]^{\frac{3}{x^2\,(1+\varepsilon)}}\right\}^{\frac{1+\varepsilon}{3\,x^{n-2}}} = e^{\underset{x=0}{\text{Lim.}}\ \frac{1+\varepsilon}{3x^{n-2}}}$$

$$=\left(\varepsilon^{\frac{1}{3x^{n-2}}}\right)_{x=0}.$$

This has the advantage of being a direct, closed expression, which shows at once the particular results, 1, $e^{\frac{1}{3}}$, ∞, when n equals 1, 2, $2+r$, where r is any positive integer, and also the general law for any value of n, of whatever character it may be.

MECHANICS.

73. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.

A sixteen foot plank weighs thirty-two pounds and is supported by two props, four feet and two feet from the ends. What weight is supported by each prop?

Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; J. SCHEFFER, A. M., Hagerstown, Md.; CHARLES C. CROSS, Libertytown, Md.; ELMER SCHUYLER, High Bridge, N. J.; A. H. BELL, Hillsboro, Ill.; CHARLES E. MEYERS, Canton, O., and M. A. GRUBER, A. M., War Department, Washington, D. C.

Let x=the number of pounds supported by the prop four feet from the end. Then taking moments around the other point of support, we have:

$$(6+4)x=6\times 32, \text{ or } x=19.2 \text{ pounds.}$$

The other prop supports $32-19.2=12.8$ pounds.

Also solved by *G. B. M. ZERR.*

74. Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

In the experiment of swinging in a vertical circle a glass containing water, and suspended by means of a string, if the string be two feet long, what must be the velocity at the lowest point if the experiment is to succeed? [From *Ziwet's Theoretical Mechanics*, Part III., p. 96.]

Solution by the PROPOSER.

Let M=the mass of the glass and water, f=the centrifugal force, r=2 feet =radius. Then $f=Mv^2/r$.

At the highest point, in order that the experiment may be successful $f=Mg$. $\therefore\ Mv^2/r=Mg$, whence $v=\sqrt{rg}$.

The velocity, v_1, due to gravity in passing from the highest to the lowest point is $v_1=\sqrt{(2gs)}$, where $s=2r$.

Hence, the velocity, v_2, at the lowest point is,

$$v_2=\sqrt{(v^2+v_1^2)}=\sqrt{(rg+2gs)}=\sqrt{(321.6)}=17.94 \text{ feet per second,}$$

g being equal to 32.16.

Also solved with slightly different results by *G. B. M. ZERR*, and *CHARLES E. MEYERS.*

DIOPHANTINE ANALYSIS.

70. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

Give methods for decomposing numbers into squares, cubes, or biquadates and show that 61×200^3 is the sum of ten cube numbers and that 844933 is the sum of eleven biquadrates in thirteen different ways. [From *The Mathematical Magazine*, Vol. II, No. 10.]

Solution by the PROPOSER.

Let s_1=sum of products taking 1 number at a time ....(1),
s_2=sum of products taking 2 numbers at a time................(2),
s_3=sum of products taking 3 numbers at a time................(3),
....
s_n=sum of products taking n numbers at a time................(n).

Then $a+b+c+d\ldots\ldots n=s_1$ ....(b_1),

$(1)(b_1)-2(2)$ gives

$a^2+b^2+c^2+d^2 \ldots\ldots n^2=s_1^2-2s_2$ ....(b_2),

$(1)(b_2)-[(2)(b_1)-3(3)]$ gives

$a^3+b^3+c^3+d^3\ldots\ldots n^3=s_1^3-3s_1s_2+3s_3$ ....(b_3),

$(1)(b_3)-[(2)(b_2)-(3)(b_1)-4(4)]$ gives

$a^4+b^4+c^4+d^4\ldots\ldots n^4=s_1^4-4s_1^2s_2+4s_1s_3+2s_2^2+4s_4$ ....(b_4),

$(1)(b_4)-[(2)(b_3)-(3)(b_2)-(4)(b_1)-5(5)]$ gives

$a^5+b^5+c^5+d^5\ldots\ldots n^5=s_1^5-5s_1^3s_2+5s_1^2s_3+5s_1s_2^2+5s_1s_4-5s_2s_3+5s_5$ ..(b_5),

etc. etc. etc.

[In (b_5) let $s_1=x$, $s_2=-xy$, $s_3=2xy^2$, $s_4=y^2(x^2-y^2)$, and $s_5=\frac{1}{5}y^5$, then $a^5+b^5+c^5+d^5+e^5+f^5=(x+y)^5$. This answers Dr. Drummond's note on page 182, Vol. V., of the MONTHLY.]

71. Proposed by A. H. BELL, Hillsboro, Ill.

Find five numbers such that the product of any two plus 1 will equal a square.

I. Partial Solution by CHARLES C. CROSS, Libertytown, Md.

We readily find four numbers to be $(n-1)$, $(n+1)$, $4n$, and $4n(4n^2-1)$.

Let n=2, 3, 4, and so on, and we get

1,	3,	8,	120,	also	1,	8,	15,	528,
2,	4,	12,	420,		2,	12,	24,	2380,
3,	5,	16,	1008,		3,	16,	33,	6440,
4,	6,	20,	1980,		4,	20,	42,	13572,
etc., etc.					etc., etc.			

From which we get a more general formula for four numbers and find it to be, m, $n^2-1+(m-1)(n-1)^2$, $n(mn+2)$, $m(m^2n-2mn+mn^2+4n-2)^2+(m^2n-2mn+mn^2+4n-2)$.

If there can be found a value for n that will render $(n-1)(128n^3-8n-8)(256n^6-256n^4+63n^2-2n-1)+(256n^6-288n^4-32n^3+65n^2+2n+1)^2=\square$ then the five numbers are $(n-1)$, $(n+1)$, $4n$, $4n(4n^2-1)$, and $[(256n^6-256n^4+63n^2-2n-1)(128n^3-8n-8)]/(256n^6-288n^4-32n^3+65n^2+2n+1)^2$; but life is too short to attempt this.

Dr. Zerr says, "I have not been able to find the five numbers."

If you get no answers to problem 71, I can give Legendre's (improved), and it is a fractional answer for the fifth number. Mr. Wilkes has found four numbers, as follows: $n-1$, $n+1$, $4n$, $16n^3-4n$. Taking these for four of them then I can prove that if there is another integral number it must end in 0.

I have investigated, up to numbers having over five thousand digits, without finding any to answer. A. H. BELL.

I have given considerable time to problem 71, but as yet have failed to obtain five numbers. I have found general values for three such numbers, viz., m, $n(mn\pm2)$, and $(n+1)[(n+1)m\pm2]$. I have also found partial general values for four such numbers, viz., m, $m\pm2$, $4(m\pm1)$, and $16(m\pm1)^3-4(m\pm1)$; but one of the sets of four numbers that I obtained by inspection, can not be obtained by this formula. The set is 1, 8, 15, 528. M. A. GRUBER.

MISCELLANEOUS.

63. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Show that the path of a projectile moving with a constant velocity is an inverted catenary of equal strength.

Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

The differential equations are, obviously,

$$\frac{d^2y}{dt^2}=-g,\quad \frac{ds}{dt}=c.$$

Whence, $\dfrac{dy}{dt}=-gt+k$, $\dfrac{dx}{dt}=\sqrt{c^2-(gt-k)^2}$, $x+k'=-\dfrac{1}{g}\cos^{-1}\dfrac{gt-k}{c}$.

By division, $$\frac{dy}{dx}=\frac{-\cos g(x+k')}{\sqrt{c^2-c^2\cos^2 g(x+k')}}=-\cot g(x+k'),$$

$$\text{or, } \cot^{-1}\frac{dy}{dx}=-g(x+k').$$

Differentiation gives

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-1}\frac{dy^2}{dx^2}=g.$$

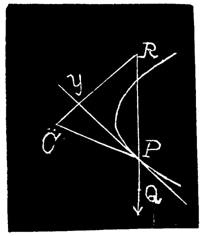
This is the differential equation of the catenary of uniform strength. See Routh's Statics, page 329, or Minchin's Statics, page 315.

64. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

Find the caustic by reflection of an hyperbola, the bright point being the center.

Solution by the PROPOSER.

Let RPQ be a reflected ray, Q a point of the caustic where RPQ touches the caustic, R the corresponding point on the secondary caustic. Then RQ is the radius of curvature of the secondary caustic, that is, $RQ=2\rho$, where ρ is the radius of curvature of the locus of y, the foot of the perpendicular from the center C on the tangent at P. Let δ be the perpendicular from C on the tangent at y to the first pedal, then $\delta r=p^2$ (Williamson's Differential Calculus, Art. 188, page 228, Sixth Edition).



Differentiating, $r\dfrac{d\delta}{dp}+\delta\dfrac{dr}{dp}=2p$. But $\rho=\dfrac{pdp}{d\delta}$; and since $p^2(r^2-a^2+b^2)=a^2b^2$ from the hyperbola.

$$\therefore\ \frac{rdr}{dp}=-\frac{a^2b^2}{p^3}.\quad \therefore\ \frac{r}{\rho}-\frac{p^2}{r}\cdot\frac{a^2b^2}{p^4r}=2,\ \text{or}\ \frac{a^2b^2}{p^2r^2}=\frac{r}{\rho}-2.$$

Let (m, n) be the coördinates of R, ψ the eccentric angle of P, (x, y) the coördinates of Q. Then

$$\frac{QR}{QP}=\frac{x-m}{x-a\sec\psi}=\frac{y+n}{y-b\tan\psi}=\frac{2\rho}{2\rho-r}.$$

In triangle CPR, $Cy=Ry$, $CP=RP=r$.

$$\therefore\ \frac{x-m}{x-a\sec\psi}=\frac{y+n}{y-b\tan\psi}=-\frac{2p^2r^2}{a^2b^2}=\frac{2r^2}{a^2-r^2-b^2}.$$

$$\therefore\ x(a^2-b^2-3r^2)=m(a^2-b^2-r^2)-2r^2a\sec\psi.$$

$$y(a^2-b^2-3r^2)+n(a^2-b^2-r^2)+2r^2b\tan\psi=0.$$

Now $m^2+n^2=4p^2$, $(m-a\sec\psi)^2+(n+b\tan\psi)^2=r^2$.

$$\therefore\ m=\frac{2ab^2\sec\psi}{a^2\tan^2\psi+b^2\sec^2\psi},\quad n=\frac{2a^2b\tan\psi}{a^2\tan^2\psi+b^2\sec^2\psi}.$$

But $m(a^2-b^2-r^2)=-m(r^2-a^2+b^2)=-m(a^2\tan^2\psi+b^2\sec^2\psi)=-2ab^2\sec\psi$.

$\therefore x(a^2-b^2-3r^2)=-2a\sec\psi(r^2+b^2)=-2a\sec^3\psi(a^2+b^2)$.

$y(a^2-b^2-3r^2)=-2b\tan\psi(r^2-a^2)=-2b\tan^3\psi(a^2+b^2)$.

By division, $\sin\psi=(y/b)^{\frac{1}{3}} / (x/a)^{\frac{1}{3}}$.

Eliminating ψ between the last two equations,

$$[(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}](a^2-b^2-3r^2)^{\frac{2}{3}}=[2(a^2+b^2)]^{\frac{2}{3}}.$$

$$\therefore [(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}]^{\frac{3}{2}}(a^2-b^2-3r^2)=2(a^2+b^2).$$

Now $r^2=a^2\sec^2\psi+b^2\tan^2\psi=\dfrac{a^2(x/a)^{\frac{2}{3}}+b^2(y/b)^{\frac{2}{3}}}{(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}}$.

$$\therefore [(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}]^{\frac{1}{2}}\{(a^2-b^2)[(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}]-3[a^2(x/a)^{\frac{2}{3}}+b^2(y/b)^{\frac{2}{3}}]\}, =2(a^2+b^2).$$

or $[(x/a)^{\frac{2}{3}}-(y/b)^{\frac{2}{3}}]^{\frac{1}{2}}[(y/b)^{\frac{2}{3}}(\frac{1}{2}a^2-b^2)-(x/a)^{\frac{2}{3}}(a^2+\frac{1}{2}b^2)]=a^2+b^2$, the caustic required.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

104. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

If I should buy goods at a price 20% higher than I did buy them, and sell the goods for the same amount that I did sell them, I would gain 25% less than I did gain. What per cent. did I gain? (Solve by Arithmetic).

105. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

A teacher looks at his watch when leaving school at noon. When he comes back he finds that the hour hand and the minute hand had just changed places (that they had when he left the school). What time was it when he left, and what time when he came back to school? (Solve by Arithmetic).

*** Solutions of these problems should be sent to B. F. Finkel not later than February 10.

GEOMETRY.

111. Proposed by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Given that the area of a triangle is equal to half the product of two sides and the sine of the included angle, prove that $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

112. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

The tangent planes at A, B, C, D to the sphere circumscribing the tetrahedron $ABCD$ form a tetrahedron $abcd$; prove that Aa, Bb, Cc, Dd will meet in a point if $BC.AD=CA.BD=AB.CD$.

113. **Proposed by T. W. PALMER, Professor of Mathematics, University of Alabama, University, Alabama.**

Given three concentric circles. Draw a straight line from the inner to the outer circumference that shall be bisected by the middle circumference.

*** Solutions of these problems should be sent to B. F. Finkel not later than February 10.

MECHANICS.

79. **Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.**

The four wheels of a street car are rigidly fixed to their axles so that axles and wheels turn together. Is it more advantageous to apply the brakes to the front or to the rear wheels, supposing the brakes to block the wheels in each case ?

80. **Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

A circular board is placed on a smooth horizontal plane and a boy runs with uniform speed around on the board close to its edge. Find the motion of the center of the board.

81. **Proposed by JAMES S. STEVENS, Professor of Physics, The University of Maine, Orono, Me.**

Two iron spheres whose weights are a and b and a is greater than b, are suspended over a frictionless pulley so that they move in a liquid medium of density δ. Assume that the density of the iron is δ', what would be the spaces passed over (downward by a and upward by b) in the first four seconds, if the spheres start from rest ?

*** Solutions of these problems should be sent to B. F. Finkel not later than February 10.

EDITORIALS.

We take this opportunity to express our thanks to our valued contributor, Prof. J. Scheffer, for preparing the index to this volume of the MONTHLY.

Prof. Edgar W. Bass, author of *Elements of Differential Calculus*, and Professor of Mathematics in the United States Military Academy at West Point for the past twenty years, has been placed on the retired list.

This issue concludes the fifth volume of the MONTHLY. The benefit the MONTHLY has done for the progress and advancement of mathematics during the past year must be told by others, but from the many enthusiastic testimonials we have received during the course of the year, the inference is that it is quite material. A number of important articles by some of our foremost mathematicians have appeared during the year. These articles are of the highest interest and value, dealing as they do with the more recent discoveries in mathematics. It is to be hoped that a larger number of contributors in the new fields of mathematical investigation will be added to those who are now carrying on this work ; for we believe that no other mathematical journal in this country offers such a great opportunity to reach the large body of teachers of mathematics as does the MONTHLY.

For example, *The American Journal of Mathematics* has a very limited constituency in this country, the reason being that it deals with only the highest subjects of mathematics and the treatment of these subjects are too abstruse and recondite to be read by any but the most advanced students of mathematics. *The Annals of Mathematics* being of a slightly less advanced character appeals to a larger body of readers, and therefore exerts a wider *influence* on the development of mathematics among us.

Five years ago, it seemed that the time had arrived when another mathematical journal could be started, which would draw to its support all persons interested in the unraveling of intricate problems in the various branches of mathematical as well as those who had gone far beyond this stage into the field of original research. It was believed then, and five years experience in the work has confirmed the belief, that a journal devoting about half its space to the solution of problems, and the other half to mathematical papers by acknowledged authorities in the various lines of original investigation, would be of great value. In a few cases we have published some papers from considerations other than scientific. We admit that such considerations are unfortunate, but in the editing of a scientific magazine, there are not unfrequently problems proposed more difficult of solution than merely deciding as to the merits of any particular article for publication. The editor of the MONTHLY has had many of these to solve. However, the appearance of unscientific articles is very rapidly diminishing, and we hope that none will appear in the future.

An apology for a department of Problems and Solutions in the MONTHLY may be found in Professor Elliott's address before the London Mathematical Society on November 10. Among other things, he says, "Secondary work is necessary that the transition from narrow to widened views of Mathematical opportunity be effected surely and without discouragement. The passion among us for examination into elegant incidentals, which shows itself in the fascination exercised by problem making and solving, must be reckoned with, and in my opinion not discouraged. Unambitious work of definitely educational intention in subjects now made known to the select few by ambitious treatises is needed.

Unassuming partial and introductory books of didactic character on modern subjects are wanted." We make mention of this, because our best mathematicians have treated too lightly the importance of well chosen problems as means to stimulate the inquirer to press his investigations into the vaster regions beyond the stage of mere problem solving. If those who are sitting in the dazzling light of truth would only condescend to hand down some of the truths so very clear to them in a form easily translated by those sitting in darkness, vast improvement in mathematical teaching would quickly follow. Only quite recently has there been a move in this direction. Men of acknowledged ability are now beginning to see that the way to improve the teaching of mathematics is to improve the works in the elementary branches. To this work, Professor Klein is now devoting much of his time, and it is to be hoped that many of the great teachers and creators of modern mathematics may follow his example.

There seems to be room for one more mathematical journal in this country. It is a journal particularly adapted to the wants of that large body of teachers of mathematics in our High Schools. What sort of a journal this body of workers want we are unable to say, but true it is that very few of them read any of the mathematical journals now published in this country. Not more than half a dozen are found on our list of subscribers while we believe that not so many are found on the lists of the *American Journal of Mathematics* and *The Annals of Mathematics* together. There is not one reason but a number of reasons why this condition exists, a discussion of which would lead us entirely beyond the limits of a mere note. But one reason is that the great majority of the positions in our high schools are filled without any special reference to the qualification of the applicant to teach mathematics, it being a prevalent opinion that any one who has the multiplication table well at hand is quite competent to teach mathematics, and thus many of our high school positions are filled by persons who are as absolutely incompetent to teach mathematics as is a person who only knows the Greek alphabet to teach Greek. We are not saying that teachers of mathematics in our high schools do not read any literature on the subject they teach. Many of them read quite a good deal, but if some of this reading were devoted to some good, live mathematical journal instead of reading the extended discussions of the "idiotic" Grube Method and kindred methods found in the ordinary educational journals, it would be better for the teacher and infinitely better for his pupils. Hence, if these positions were filled by persons who are in love with their subject, a journal of the scope of the MONTHLY might meet their wants.

If every college and university would follow the principle established by one of our largest universities, viz., to fill mathematical positions with first-class teachers of mathematics, a very necessary improvement would follow in a very short time. Not infrequently are mathematical positions filled by persons who, while in college, took an extended course in latin, or Greek, or History, and their work in mathematics not extending beyond trigonometry. This is always done to the great detriment of mathematics both as a science and as an art.

While such conditions exist, it is desirable that every opportunity available should be siezed to make improvements in the teaching of mathematics, by, as much as possible, filling positions in mathematics with good teochers of mathematics, and by improving the methods of presenting the fundamental principles of the science in our elementary text-books.

The MONTHLY offers at present the best opportunity to disseminate sound doctrine in the elementary branches, and if some of our best mathematicians would take up this work, great and lasting good would result.

We take this opportunity to thank Dr. Miller for the good work he is doing in discussing the Group Theory; to Dr. Halsted for his continued articles on Non-Euclidean Geometry; to Dr. Roe for his articles on Symmetric Functions; and to all others who have contributed articles, problems, and solutions. We shall rely on your continued support during the coming year, and by united effort much may be accomplished to create a larger interest in mathematics.

Efforts are being made to make the MONTHLY a permanent factor in the development and progress of mathematics in the United States, either by endowing it so that it may remain where it is now or by transferring it to some university where ample provision will be made for its perpetual maintenance.

BOOKS AND PERIODICALS.

A Treatise on Roofs and Bridges with Numerous Exercises. By Edward A. Bowser, LL. D., Professor of Mathematics and Engineering in Rutgers College. 8vo. Cloth. 195 pages. New York: D. Van Nostrand Co.

This little work is in line with the other works written by Dr. Bowser. Logical order, simplicity of treatment even with a difficult subject are the prominent features of the work. The principles and methods employed in finding the forces in Roofs and Bridges are clearly explained. The whole subject is discussed and developed in accordance with the best methods used in the modern practice of Roof and Bridge construction.

B. F. F.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place, New York.

The question of the bearings of our federal Constitution on the government of newly acquired territories, about which so much haze seems to have gathered in the popular mind, is very clearly and exhaustively treated in the *American Monthly Review of Reviews* for January by Professor Harry Pratt Judson, of the University of Chicago. Professor Judson reaches the conclusion that the Constitution presents no difficulties whatever to our acquisition and control of such territories as the Philipines, for example.

B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson.

Julian Ralph, who has been for some years in England, writes for the December *Cosmopolitan* a very clever analysis of what seems to him the English ideas of a gentleman; and Mr. John Brisben Walker attempts to consider the American ideals. We are in the formative stage of American manners, and too much stress can scarcely be given to the dangers of introducing those ideas which are least admirable in th character of our English cousins.

B. F. F.

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